A theorem of Teichmüller, uniformly perfect sets and punishing factors

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Abstract

We prove quantitative relations with sharp constants between several quantities used to define uniformly perfect sets. They are applied to derive bounds for the second derivative of analytic functions $f : \Omega \to \Pi$, where Ω and Π are open sets in the plane $\mathbb{C}$ such that $\overline{\mathbb{C}} \setminus \Omega$ and $\overline{\mathbb{C}} \setminus \Pi$ are uniformly perfect sets.

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1. Introduction

One of the motivations for the present paper is the following remark from the book [10] of Carleson and Gamelin (see p. 64):
A simple scaling and normal families argument shows that conformal annuli of large modulus contain genuine annuli of large modulus.
As usual the modulus of a genuine annulus

$$An(r, s; z_0) = \{z \mid r < |z - z_0| < s\}$$

with center z_0 is defined by

$$M(An(r, s; z_0)) = \frac{1}{2\pi} \log \frac{s}{r},$$

whereas the modulus $M(\Omega)$ of a conformal annulus, or a doubly connected domain, equals the modulus of the genuine annulus that is conformally equivalent to Ω.
We want to give a quantitative version of the qualitative statement cited above and to compare by explicit estimates characteristics of uniformly perfect sets on the Riemann sphere.

Let $\Omega \subset \mathbb{C}$ be an open set with more than one boundary point in $\mathbb{C}$. A doubly connected domain $D \subset \Omega$ is said to separate the compact (in the usual topology on $\overline{\mathbb{C}}$) set $E = \overline{\mathbb{C}} \setminus \Omega$ if both components of $\overline{\mathbb{C}} \setminus D$ have non-empty intersections with E. A compact set $E = \overline{\mathbb{C}} \setminus \Omega$ is said to be uniformly perfect if

$$M(\Omega) = \sup\{M(D) \mid D \subset \Omega \text{ doubly connected and separating } E\}$$

is finite.
The most known quantity describing the geometry of uniformly perfect sets was defined by Pommerenke in [19].
A compact set E on the Riemann sphere $\overline{\mathbb{C}}$ that contains the point at infinity is said to be uniformly perfect if there exists a constant $c > 0$ such that, for every $z_0 \in E \setminus \{\infty\}$ and every $r \in (0, \infty)$, the set $E \cap An(cr, r; z_0)$ is not empty.
Let $C_0(E)$ be the supremum of the admissible constants for an uniformly perfect set. It is evident that

$$C_0(E) \;=\; e^{-2\pi\, M_0(\Omega)}, \quad \Omega = \mathbb{C} \setminus E,$$

where $M_0(\Omega)$ is defined as the supremum of the moduli of all genuine annuli $An \subset \Omega$ such that An separates E and the center of An belongs to $\partial\Omega$ (see [19] and also [8], [9], [10], [11], [20]). We take $M(\Omega) = 0$ for open sets that have simply connected components only and $M_0(\Omega) = 0$, if Ω contains no genuine annulus separating E.
Since any genuine annulus is a conformal annulus, the inequality

$$M_0(\Omega) \;\leq\; M(\Omega) \tag{1}$$

is trivial and the Carleson- Gamelin remark implies

$$M(\Omega) \;=\; \infty \quad \Longrightarrow \quad M_0(\Omega) \;=\; \infty. \tag{2}$$

On the other hand, using his theorem on extremal moduli that we cite as Theorem A in Section 3, Teichmüller proved in [22] that any doubly connected domain D with $M(D) > 1/2$ contains a circle which separates the components of its complement. Moreover, the center of this circle may be chosen at the end of the diameter of the bounded component of $\mathbb{C} \setminus D$.
Solynin (see [21]) and Herron, Liu, and Minda (see [13]) proved that a separating circle exists for doubly connected domains D with $M(D) > 1/4$, if one does not restrict the position of the center of the circle. In [13] it is also proved that such D contain separating annuli An of modulus $M(An) = M(D) - c(c \approx 0.46)$, where again no restriction is imposed on the center of the annulus. In the same paper, they prove that $M(An) \geq M(D) - \tilde{c}(\tilde{c} = \frac{1}{\pi}\log(2\sqrt{2} + 2) \approx 0.501)$ if one fixes the center of An on ∂D.

In the following we will use the result of Teichmüller to derive a new quantitative version of the Carleson-Gamelin remark, where we find as the best possible constant 1/2 in a similar inequality not only chosing the center of An on ∂D but fixing it as Teichmüller did.
We will apply this result to different characterizations of uniformly perfect sets and to estimates of the second derivative of functions analytic on open sets with uniformly perfect boundaries.

2. Main results

In addition to (1) and (2) we prove as Theorem 1 that

$$M(\Omega) \leq M_0(\Omega) + \frac{1}{2} \tag{3}$$

for any open set $\Omega \subset \mathbb{C}$. The proof is based on the above cited theorem of Teichmüller and on a related formula of Ahlfors in [1].
In Section 4, we consider the problem to compare $M_0(\Omega)$ with the following characteristics of the hyperbolic geometry on Ω:

$$\alpha(\Omega) = \inf\{\lambda_\Omega(z)\mathrm{dist}(z, \partial\Omega) \mid z \in \Omega\} \tag{4}$$

and

$$\gamma(\Omega) = \sup\left\{\left|\nabla\lambda_\Omega^{-1}(z)\right| \mid z \in \Omega\right\}, \tag{5}$$

where λ_Ω is the density of the Poincaré metric with curvature -4 defined on the components of Ω.
In Theorem 2 we will prove that

$$\sup\{M_0(\Omega)\alpha(\Omega)\} = \frac{1}{4} \tag{6}$$

and that

$$\sup\left\{\frac{M_0(\Omega)}{\gamma(\Omega)}\right\} = \frac{1}{4}, \tag{7}$$

where the supremum is taken with respect to all open sets $\Omega \subset \mathbb{C}$ that have more than one boundary point in $\mathbb{C}$. The assertion (6) gives the sharp form of the estimate

$$M_0(\Omega)\alpha(\Omega) \leq \frac{1}{2}$$

proved in [9]. In the proof of (6) and (7) we show that

$$\frac{(\gamma(An))^2}{16} = \frac{1}{4} + (M(An))^2$$

for any genuine annulus An and that

$$4M(An) \sim \frac{1}{\alpha(An)}$$

for genuine annuli if $M(An) \to \infty$.
Concerning the last section, we must confess that the above considerations have their origin in the following question that we discuss since long (see [3], [4], [5], [7], and in special [6]).
Let Ω and Π be open sets in $\mathbb{C}$ or in $\overline{\mathbb{C}}$ and let $A(\Omega, \Pi)$ be the set of functions $f : \Omega \to \Pi$ locally holomorphic or meromorphic and in general multivalued. What can be said on the influence of the geometric properties of Ω and Π on the quantities

$$C_n(\Omega, \Pi) \;=\; \sup\left\{ \frac{\left|f^{(n)}(z)\right|}{n!} \, \frac{\lambda_\Pi(f(z))}{(\lambda_\Omega(z))^n} \;\mid\; z \in \Omega,\; f \in A(\Omega, \Pi) \right\},$$

where all possible values of $f(z)$ are taken into account. Following a proposal of Pommerenke, we call these $C_n(\Omega, \Pi)$ punishing factors, since $\lambda_\Omega(z)$ and $\lambda_\Pi(f(z))$ may be considered as measures of the positions of z in Ω and $f(z)$ in Π in the hyperbolic geometries of Ω and Π, whereas $C_n(\Omega, \Pi)$ are factors punishing bad behavior at their boundaries. The above considerations give us the possibility to determine bounds for $C_2(\Omega, \Pi)$ in terms of the quantities $M_0(\Omega)$ and $M_0(\Pi)$ that are "visible" in Euclidean geometry.
During these proofs we get a sharp form of the Osgood-Jørgensen inequality (see [15] and [17]), namely

$$\sup\{\sup\{|\nabla \log \lambda_\Omega(z)| \mathrm{dist}(z, \partial\Omega) \mid z \in \Omega\}\} \;=\; 2, \tag{8}$$

where the first supremum is taken with respect to all hyperbolic domains $\Omega \subset \overline{\mathbb{C}}$.

3. Comparison of moduli

We are concerned with the following theorem of Teichmüller (see [22] and also [1]).

Theorem A. *Of all doubly connected domains that separate the pair* $\{-1, 0\}$ *from a pair* $\{w_0, \infty\}$ *with* $|w_0| = R$ *the one with the biggest modulus is the complement of the union of the segments* $[-1, 0]$ *and* $[R, \infty]$.

Clearly, to prove the inequality (3) for any open set Ω it is sufficient to consider the special case where Ω is a doubly connected domain. Hence, the inequality (3) is a corollary of the following extension of Teichmüller's result on the existence of a separating circle.

Theorem 1. *Any doubly connected domain* $D \subset \mathbb{C}$ with $M(D) > 1/2$ *contains an annulus* $An = An(r, s; z_0)$ *such that* $M(An) = M(D) - 1/2$, *the* *center z_0 belongs to the bounded component E_1 of $\overline{\mathbb{C}} \setminus D$ and* $\operatorname{diam} E_1 = r$. *The constant 1/2 is sharp.*

Proof of Theorem 1. Let $E = \overline{\mathbb{C}} \setminus D = E_1 \cup E_2$. Without loss of generality we may assume that the diameter of E_1 equals 1 and that the points $z = 0$ and $z = -1$ belong to E_1. Now, we consider the annulus $An(1, m; 0)$, where $m = \exp(2\pi(M(D) - 1/2))$. It has the following properties: $An(1, m; 0) \cap E_1 = \emptyset$, the center of $An(1, m; 0)$ belongs to E_1, and $M(An(1, m; 0)) = M(D) - 1/2$.
If $An(1, m; 0) \subset D$, then $An(1, m; 0)$ separates E and there is nothing to prove. Let us assume that this is not true, i. e. that there exists a point $w_0 \in E_2$ such that $1 < |w_0| < m$. Taking $|w_0| = R$ we consider the Teichmüller annulus

$$D(R) = \mathbb{C} \setminus ([-1, 0] \cap [R, \infty))$$

with the modulus $\Lambda(R)$ (compare [1]). According to Theorem A the inequality

$$M(D) \leq \Lambda(R) \tag{9}$$

is valid. On the other hand, we will prove that the assumed condition $1 < R < m$ implies that

$$\Lambda(R) < \frac{1}{2\pi} \log R + \frac{1}{2} < M(D). \tag{10}$$

Clearly, (10) contradicts (9). Hence, to complete the proof of Theorem 1 it is sufficient to show that the first inequality in (10) holds for any $R > 1$. The function $\Lambda(R)$ is implicitely defined by the following formula due to Ahlfors (see [1], p. 75)

$$R = \frac{1}{16\, q} \prod_{n=1}^{\infty} \left(\frac{1 - q^{2n-1}}{1 + q^{2n}} \right)^8, \quad q = e^{-2\pi\Lambda(R)}. \tag{11}$$

It is known that $\Lambda(1) = 1/2$ and that $\Lambda(R) \to \infty$ as $R \to \infty$. If one considers the first equation in (11) as the definition of a function $R(q)$, it is evident that $qR(q)$ is a decreasing function of q for

$$0 < q < e^{-\pi} = e^{-2\pi\Lambda(1)}.$$

The function $qR(q)$ decreases from 1/16 to $\exp(-\pi)$, when q increases from 0 to $\exp(-\pi)$. In particular,

$$16 < \frac{e^{2\pi\Lambda(R)}}{R} < e^{\pi} = 23.14\ldots.$$

for any $R > 1$, and (10) follows.
The sharpness of the constant $1/2$ is a consequence of Teichmüller's considerations, since in the case $M(D) = 1/2$ there does not exist a separating annulus with the above properties.
This completes the proof of Theorem 1.

In contrast to $M(\Omega)$ the quantity $M_0(\Omega)$ is not a conformal invariant. Nevertheless, it may be usefull to remark the following consequence of Theorem 1.

Corollary 1. *Let Ω_1 and Ω_2 be two conformally equivalent domains in $\mathbb{C}$. If $M_0(\Omega_1)$ is finite then*

$$|M_0(\Omega_1) \; - \; M_0(\Omega_2)| \le \frac{1}{2}.$$

Proof. Since $M(\Omega)$ is conformally invariant, we have that

$$M(\Omega_1) \; = \; M(\Omega_2) =: M.$$

The inequalities (1) and (3) imply

$$M_0(\Omega_1) \le \; M \; \le M_0(\Omega_2) + \frac{1}{2}.$$

Hence,

$$M_0(\Omega_1) \; - \; M_0(\Omega_2) \le \frac{1}{2}.$$

In the same way, we get

$$M_0(\Omega_2) \; - \; M_0(\Omega_1) \le \frac{1}{2},$$

which completes the proof.

Remark. For any genuine annulus An, the inequality $M_0(An) < M(An)$ holds, since we consider only separating genuine annuli in An, which have a center lying on ∂An. Geometrically it is evident that a genuine annulus An has such a separating annulus, if and only if

$$\frac{1}{2\pi} \log 3 \; < \; M(An) \; \le \; M_0(An) + \frac{1}{2\pi} \log 3$$

(see [13] and [2]).

4. Proofs of the equations (6) and (7)

In this section we consider the characteristics $\alpha(\Omega)$ and $\gamma(\Omega)$ defined in (4) and (5) for open sets $\Omega \subset \mathbb{C}$ with more than one boundary point in $\mathbb{C}$ and we compare them with $M_0(\Omega)$. Firstly, we cite some known results. In [17], Osgood proved that

$$\frac{1}{\alpha(\Omega)} \leq \gamma(\Omega) \leq \frac{2}{\alpha(\Omega)}, \quad \Omega \subset \mathbb{C}. \tag{12}$$

In [9], the first paper on uniformly perfect sets, Beardon and Pommerenke got the inequality

$$\frac{1}{\alpha(\Omega)} \leq 2\pi\, M_0(\Omega) + 2K, \quad \Omega \subset \mathbb{C},$$

where K is the constant from Landau's theorem. Nowadays, it is known from the sharp form of this theorem (see [12] and [14]) that

$$K = \frac{(\Gamma(1/4))^{1/4}}{(2\pi)^2} \approx 4.38.$$

In [9], it is also proved that

$$M_0(\Omega) \leq \frac{1}{2\alpha(\Omega)}, \quad \Omega \subset \mathbb{C}. \tag{13}$$

The following theorem assures that the equations (6) and (7) hold, where (6) is the sharp form of (13).

Theorem 2. *Let $\Omega \subset \mathbb{C}$ be an open set with more than one boundary point in $\mathbb{C}$. If $\alpha(\Omega) > 0$, then the inequalities*

$$M_0(\Omega) < \frac{\gamma(\Omega)}{4} \quad \textit{and} \quad M_0(\Omega) < \frac{1}{4\alpha(\Omega)} \tag{14}$$

are valid. The constant $1/4$ is best possible in both cases because of the equations

$$\lim_{\epsilon\to 0+} \frac{M_0(An(\epsilon,1;0))}{\gamma(An(\epsilon,1;0))} = \lim_{\epsilon\to 0+} M_0(An(\epsilon,1;0))\alpha(An(\epsilon,1;0)) = \frac{1}{4}. \tag{15}$$

Proof. We first prove (14). To that end, it is sufficient to consider the cases where $0 < M_0(\Omega) < \infty, \gamma(\Omega) < \infty$, and $\alpha(\Omega) > 0$. We have to estimate the modulus $M(An)$ for any genuine annulus $An := An(r_1, r_2; a) \subset \Omega$ such that (i) $0 < r_1 < r_2 < \infty$,

(ii) $a \in \partial\Omega$,
(iii) $\{z \mid |z-a| = r_1\} \cup \partial\Omega \neq \emptyset$,
(iv) $\{z \mid |z-a| = r_2\} \cup \partial\Omega \neq \emptyset$.
We choose a point $b \in \partial\Omega$ with the property $|a-b| = r_1$ and define $z_0 \in An$ by

$$z_0 \;=\; a + \frac{b-a}{|b-a|}\, r_0, \;\; r_0 = \sqrt{r_1 r_2}.$$

We have

$$\mathrm{dist}(z_0, \partial\Omega) \;\leq\; r_0 - r_1 \;=\; r_0 \left(1 - e^{-\pi M(An)}\right)$$

and as in [9], p. 479, we get using the comparison principle (see [1], p. 53) for $An \subset \Omega$

$$\lambda_\Omega(z_0) \;\leq\; \lambda_{An}(z_0) \;=\; \frac{1}{4 r_0 M(An)}.$$

The two last formulae imply

$$\alpha(\Omega) \;\leq\; \lambda_\Omega(z_0)\mathrm{dist}(z_0, \partial\Omega) \;\leq\; \frac{1 - e^{-\pi M(An)}}{4M(An)}.$$

Using the fact that $M_0(\Omega)$ is defined as the supremum of the moduli for the genuine annuli An defined as above and the first inequality in (12), we obtain

$$\frac{M_0(\Omega)}{1 - e^{-\pi M_0(\Omega)}} \;\leq\; \frac{1}{4\alpha(\Omega)} \;\leq\; \frac{\gamma(\Omega)}{4}. \tag{16}$$

Obviously, (16) implies (14).
As a preparation of the proof of the equations (15) we first derive an interesting formula relating $M(An)$ and $\gamma(An)$ for genuine annuli An.

Lemma 1. *For any genuine annulus An the equation*

$$\frac{(\gamma(An))^2}{16} \;=\; \frac{1}{4} \;+\; (M(An))^2 \tag{17}$$

is valid.

Proof of Lemma 1. Since the conformal modulus and $|\nabla(1/\lambda_{An}(z))|$ are invariant under linear transformations of open sets, it is sufficient to consider annuli $An = An(\epsilon, 1; 0), \epsilon > 0$. Denoting by M the modulus of such an annulus, i. e.

$$M \;:=\; M(An) \;=\; \frac{1}{2\pi}\log(1/\epsilon),$$

we get

$$\frac{1}{\lambda_{An}(z)} \;=\; 4\, M\, |z| \,\sin\left(\frac{\log(1/|z|)}{2\, M}\right), \quad z \in An,$$

(see, for instances, [9]). Hence,

$$|\nabla(1/\lambda_{An}(z))| \;=\; \left|\frac{d\,(1/\lambda_{An}(z))}{d\,|z|}\right| \;=:\; |s(t)|,$$

where

$$s(t) \;=\; 4\,M\,\sin t \;-\; 2\,\sin t, \quad \text{with} \quad t = \frac{\log(1/|z|)}{2\,M} \in (0,\pi).$$

By a straightforward calculation we derive

$$\gamma(An) = \max\{|s(t)| \mid t \in [0,\pi]\} = |s(t_0)| = 2\sqrt{1+4M^2},\ \tan\, t_0 = -2M,$$

which completes the proof of Lemma 1.
The formulae (14), (17) and Theorem 1 immediately imply

$$\lim_{\epsilon\to 0+} \frac{M_0(An(\epsilon,1;0))}{\gamma(An(\epsilon,1;0))} \;=\; \frac{1}{4}. \tag{18}$$

From the second inequality of (14) we see that

$$\overline{\lim}_{\epsilon\to 0+} M_0(An(\epsilon,1;0))\alpha(An(\epsilon,1;0)) \;\leq\; \frac{1}{4}.$$

Now, let us assume that the second equation in (15) is not true, i. e.

$$\underline{\lim}_{\epsilon\to 0+} M_0(An(\epsilon,1;0))\alpha(An(\epsilon,1;0)) \;<\; \frac{1}{4}.$$

Applying once more Osgood's estimate $\alpha(\Omega) \geq 1/\gamma(\Omega)$ from (12) to $\Omega = An$ in the assumed inequality, we get

$$\underline{\lim}_{\epsilon\to 0+} \frac{M_0(An(\epsilon,1;0))}{\gamma(An(\epsilon,1;0))} \;<\; \frac{1}{4},$$

which contradicts (18).
The proof of Theorem 2 is complete.

5. An application to Schwarz-Pick type inequalities

Let Ω and Π be open sets on the Riemann sphere $\overline{\mathbb{C}}$ that are equipped with the Poincaré metric of curvature -4. According to Poincaré's generalization of Riemann's mapping theorem this means that the boundaries of Ω and Π contain at least three points in $\overline{\mathbb{C}}$ and that the density of this metric in the unit disc $\Delta = \{z \mid |z| < 1\}$ is defined as

$$\lambda_\Delta(z) \;=\; \frac{1}{1-|z|^2}, \quad z \in \Delta.$$

We consider the set $A(\Omega,\Pi)$ of functions $f:\Omega\to\Pi$, which are locally holomorphic or meromorphic and in general multivalued. Let $n\in\mathbb{N}$ and $M_n(z,w,\Omega,\Pi)$ be defined as the smallest possible value such that the inequality

$$\frac{\left|f^{(n)}(z)\right|}{n!} \le M_n(z,w,\Omega,\Pi)\frac{(\lambda_\Omega(z))^n}{\lambda_\Pi(w)}$$

holds for all $f\in A(\Omega,\Pi), f(z)=w$, and

$$C_n(\Omega,\Pi)=\sup\{M_n(z,w,\Omega,\Pi)\ |\ (z,w)\in\Omega\times\Pi\}.$$

The principle of the hyperbolic metric implies that $M_1(z,w,\Omega,\Pi)=1$, and, in turn, that $C_1(\Omega,\Pi)=1$ for any pair (Ω,Π) equipped with the above Poincaré metric. We have considered the cases $n\ge 2$ for several special cases in [3] - [7]. Especially in [3] the dependence of M_n on the hyperbolic characteristics

$$(\lambda_\Omega(z))^{-k}\frac{\partial^k\log(\lambda_\Omega(z))}{\partial\overline{z}^k} \text{ and } (\lambda_\Pi(w))^{-k}\frac{\partial^k\log(\lambda_\Pi(w))}{\partial\overline{w}^k},\ \ 1\le k\le n-1,$$

was used. In our recent paper [6] we have proved that $C_n(\Omega,\Pi)<\infty$ for a pair (Ω,Π) such that Ω and Π are finitely connected domains in $\mathbb{C}$ if and only if $\overline{\mathbb{C}}\setminus\Omega$ and $\overline{\mathbb{C}}\setminus\Pi$ are uniformly perfect.
In the following theorem we describe $M_2(z,w,\Omega,\Pi)$ in terms of the gradients

$$\nabla\left(1/\lambda_\Omega(z)\right) = -2\left(\lambda_\Omega(z)\right)^{-1}\frac{\partial\log(\lambda_\Omega(z))}{\partial\overline{z}}$$

and

$$\nabla\left(1/\lambda_\Pi(w)\right) = -2\left(\lambda_\Pi(w)\right)^{-1}\frac{\partial\log(\lambda_\Pi(w))}{\partial\overline{w}}.$$

Here we shall need the function F defined as

$$F(p,q) := \begin{cases} p+q, & \text{if } p+2q\ge 4,\\ 2+\frac{p^2}{8-4q}, & \text{if } p+2q<4,\end{cases} \tag{19}$$

for $p\ge 0$ and $q\ge 0$.

Theorem 3. *Let Ω and Π be open sets on the Riemann sphere that are equipped with the Poincaré metric of curvature -4. Then for any $f\in A(\Omega,\Pi)$ the estimate*

$$|f''(z)| \le F(p,q)\frac{(\lambda_\Omega(z))^2}{\lambda_\Pi(w)} \tag{20}$$

is valid for $w=f(z), z\in\Omega'=\mathbb{C}\cap\Omega, w\in\Pi'=\mathbb{C}\cap\Pi, p=|\nabla(1/\lambda_\Omega(z))|$, and $q=|\nabla(1/\lambda_\Pi(w))|$. The inequality (20) is sharp for any $(z,w)\in\Omega'\times\Pi'$.

Proof. We fix $(z,w) \in \Omega' \times \Pi'$ and consider a function $f \in A(\Omega,\Pi), f(z) = w$. Without loss of generality, we assume that Ω and Π equal their components that contain z and w, respectively. Let $\Phi : \Delta \to \Omega$ and $\Psi : \Delta \to \Pi$ be universal covering maps such that $\Phi(0) = z$ and $\Psi(0) = w$. We consider the holomorphic function $g : \Delta \to \Delta$ defined by

$$g(\zeta) \;=\; \Psi^{-1}(f(\Phi(\zeta))) = \sum_{n=1}^{\infty} c_n \zeta^n, \quad \zeta \in \Delta.$$

By direct computations, we get the identity

$$\frac{f''(z)}{2}\,\frac{(\Phi'(0))^2}{\Psi'(0)} \;=\; c_2 \,+\, \frac{\Psi''(0)}{2\Psi'(0)}\,c_1^2 \,-\, \frac{\Phi''(0)}{2\Phi'(0)}\,c_1,$$

and therefore

$$\frac{|f''(z)|}{2}\,\frac{\lambda_\Pi(w)}{(\lambda_\Omega(z))^2} \;=\; \left|c_2 + \frac{y}{2}\,c_1^2 + \frac{x}{2}\,c_1\right|, \tag{21}$$

where

$$x \;=\; \frac{\Phi''(0)}{\Phi'(0)} \quad \text{and} \quad y \;=\; \frac{\Psi''(0)}{\Psi'(0)}.$$

If $g \in A(\Delta,\Delta)$ and $g(0) = 0$, then the function $f = \Psi \circ g \circ \Phi^{-1}$ belongs to the set $A(\Omega,\Pi)$ and $f(z) = w$. Therefore, we have to find

$$S(x,y) \;=\; \max\left\{\left|c_2 + \frac{y}{2}\,c_1^2 + \frac{x}{2}\,c_1\right| \;\mid\; g \in A(\Delta,\Delta) \text{ with } g(0) = 0\right\}. \tag{22}$$

Using classical results on the Taylor coefficients of unimodular bounded functions we get

$$S(x,y) \;=\; \max\left\{\left|c_2 + \frac{y}{2}\,c_1^2 + \frac{x}{2}\,c_1\right| \;\mid\; |c_1| \le 1,\, |c_2| \le 1 - |c_1|^2\right\}$$

and, by a little analysis

$$S(x,y)) \;=\; \max\left\{1 - t^2 + \frac{|y|}{2}\,t^2 + \frac{|x|}{2}\,t \;\mid\; t \in [0,1]\right\} \;=\; \frac{1}{2}\,F(|x|,|y|),$$

where F is defined as in (19). Combining this with (21) and (22) we get

$$\max\{|f''(z)| \mid f \in A(\Omega,\Pi) \text{ with fixed } w = f(z)\} \;=\; F(|x|,|y|)\,\frac{(\lambda_\Omega(z))^2}{\lambda_\Pi(w)}.$$

This implies the assertion of Theorem 3, since

$$|x| = p = |\nabla\,(1/\lambda_\Omega(z))| \text{ and } |y| = q = |\nabla\,(1/\lambda_\Pi(w))|\,.$$

The theorem 3 and the content of the section 3 immediately imply the following generalization of the above cited result from [6] for $n = 2$.

Corollary 2. *Let Ω and Π be open sets in $\mathbb{C}$ equipped with the usual Poincaré metric. Then $C_2(\Omega,\Pi)$ is finite if and only if $\overline{\mathbb{C}}\setminus\Omega$ and $\overline{\mathbb{C}}\setminus\Pi$ are uniformly perfect.*

In the next corollary we want to give some bounds for $C_2(\Omega,\Pi)$ dependent on the Euclidean geometry and on the conformal geometry of these sets.

Corollary 3. *Let Ω and Π be open sets in $\mathbb{C}$ equipped with the usual Poincaré metric. Then the following assertions are valid.*

(i) $C_2(\Omega,\Pi) \;=\; \frac{1}{2}(\gamma(\Omega)+\gamma(\Pi))$,

(ii) $2(M_0(\Omega)+M_0(\Pi)) \;\le\; C_2(\Omega,\Pi) \;\le\; 2\pi(M_0(\Omega)+M_0(\Pi))+4K$,

(iii) $2(M(\Omega)+M(\Pi))-2 \;\le\; C_2(\Omega,\Pi) \;\le\; 2\pi(M(\Omega)+M(\Pi))+4K$,

(iv) $C_2(\Omega,\Pi) \;\ge\; 2$, *where equality is attained if and only if all components of Ω and Π are convex.*

Proof. For a domain Ω, it is known that $\gamma(\Omega)=2$ if and only if Ω is convex and that $\gamma(\Omega)>2$ in all other cases (see [18], [17], and [23]). Hence, (iv) follows from (i). Further, the contents of the sections 2 and 3 together with (i) imply (ii) and (iii). So, it remains to prove (i). To this end, we remark that, for $a\ge 2$ and $b\ge 2$, one easily gets the identity

$$\max\{F(p,q)\mid p\in[0,a],\text{ and } q\in[0,b]\} \;=\; a+b,$$

since the condition $p+2q<4$ implies the following chain of inequalities

$$2+\frac{p^2}{8-4q} \;<\; 2+\frac{p}{2} \;<\; a+b.$$

This completes the proof of Corollary 3.

Together with the results of [7], where we proved that $C_n(\Omega,\Pi)=2^{n-1}$ for convex domains Ω and Π, Corollary 3 (iv) delivers the proof of a weak form of a conjecture published in [3].

Theorem 4. *$C_n(\Omega,\Pi)=2^{n-1}$ for all $n\in\mathbb{N}\setminus\{1\}$ if and only if all components of Ω and Π are convex domains.*

We want to formulate the remaining parts of a stronger conjecture as

Conjecture 1. *Let $n\in\mathbb{N}\setminus\{1,2\}$ and let Ω and Π be open sets in $\mathbb{C}$ equipped with the usual Poincaré metric. Then $C_n(\Omega,\Pi)=2^{n-1}$ implies that all components of Ω and Π are convex domains.*

Now, we turn to the cases, where the point at infinity belongs to Ω or Π and we prepare these consideration with a lemma which may deserve some interest of its own.

Lemma 2. *Let Ω be an open set in $\overline{\mathbb{C}}$ equipped with the usual Poincaré metric and let $\infty \in \Omega$. Then the equations*

$$\text{(i)} \lim_{z\to\infty} |\nabla\,(1/\lambda_\Omega(z))| \;=\; \infty,$$

$$\text{(ii)} \lim_{z\to\infty} |\nabla\,\log\lambda_\Omega(z)|\mathrm{dist}(z,\partial\Omega) \;=\; 2,$$

are valid.

Proof. Without restriction of generality we assume Ω to be a hyperbolic domain. To prove (i) we consider a universal covering map f of Δ onto Ω such that $f(0)=\infty$. Let f have the expansion

$$z \;=\; f(\zeta) \;=\; \frac{a}{\zeta} + \varphi(\zeta), \quad \zeta\in\Delta,$$

where $a\neq 0$ and φ is a function meromorphic in Δ and holomorphic in a neighbourhood of the origin. Direct computations using

$$\lambda_\Omega(z) \;=\; \frac{1}{|f'(\zeta)|(1-|\zeta|^2)}, \quad z=f(\zeta), \tag{23}$$

imply

$$|\nabla\,(1/\lambda_\Omega(z))| \;=\; \frac{1}{|\zeta|}\left|\frac{2a+\zeta^3\varphi''(\zeta)}{a-\zeta^2\varphi'(\zeta)} + 2\,|\zeta|^2\right|. \tag{24}$$

If we take the limit $z\to\infty$ of the left side of equation (24), which is equivalent to taking the limit $\zeta\to 0$ of the right side, we see that (i) is valid.
To prove (ii) we multiply the left sides and the right sides of (23) and (24) with one another. If we multiply these products with $|z|$ and $|a/\zeta|$, we get

$$\lim_{z\to\infty}|\nabla\,\log\lambda_\Omega(z)||z| \;=\; \lim_{\zeta\to 0}|\nabla\,\log\lambda_\Omega(f(\zeta))|\left|\frac{a}{\zeta}\right| \;=\; 2. \tag{25}$$

Since $\partial\Omega$ is bounded,

$$\lim_{z\to\infty}\frac{\mathrm{dist}(z,\partial\Omega)}{|z|} \;=\; 1. \tag{26}$$

The formulae (25) and (26) imply (ii). This completes the proof of Lemma 2.

Corollary 3 and Lemma 2 (i) together imply

Corollary 4. *$C_2(\Omega, \Pi)$ is finite if and only if Ω and Π are open sets in $\mathbb{C}$ equipped with the usual Poincaré metric such that $\overline{\mathbb{C}} \setminus \Omega$ and $\overline{\mathbb{C}} \setminus \Pi$ are uniformly perfect.*

Lemma 2 (ii) gives rise to a sharp infinity version of the Osgood-Jørgensen theorem. For hyperbolic domains $\Omega \subset \mathbb{C}$ the fundamental inequality

$$|\nabla \log \lambda_\Omega(z)| \mathrm{dist}(z, \partial\Omega) \leq 2, \quad z \in \Omega, \tag{27}$$

was discovered by Osgood (see [17], and compare also [16]). The proof uses certain results of Jørgensen (see [15]). One may observe that this proof in [17] and the related results in [15] do not use the fact that $\infty \notin \Omega$. Moreover, in [15] Jørgensen considers the case when $\infty \in \Omega$ as a basic one. This observation together with Lemma 2 (ii) leds us to the following theorem which proves (8).

Theorem 5. *Let Ω be an open set in $\overline{\mathbb{C}}$ equipped with the usual Poincaré metric. For any $z \in \Omega \cap \mathbb{C}$ the inequality* (27) *is valid. The constant* 2 *in* (27) *cannot be replaced by a smaller one if $\infty \in \Omega$.*

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