INTEGRAL MEANS AND DIRICHLET INTEGRAL FOR ANALYTIC FUNCTIONS

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Abstract. For normalized analytic functions f in the unit disk, the estimate of the integral means

$$L_1(r,f) := \frac{r^2}{2\pi}\int_{-\pi}^{\pi} \frac{d\theta}{|f(re^{i\theta})|^2}$$

is important in certain problems in fluid dynamics, especially when the functions $f(z)$ are non-vanishing in the punctured unit disk $0<|z|<1$. We consider the problem of finding the extremal function f which maximizes the integral means $L_1(r,f)$. In addition, for certain class $\mathcal{F}$ of analytic functions, we solve the extremal problem for the Yamashita functional

$$A(r) = \max_{f\in\mathcal{F}} \Delta\left(r, \frac{z}{f(z)}\right) \quad \text{for } 0<r\leq 1.$$

1. Preliminaries and Main Results

Denote by $\mathcal{H}$ the family of all functions f which are analytic in the unit disk $\mathbb{D} := \{z\in\mathbb{C}:\ |z|<1\}$ and by $\mathcal{A}$ the subfamily of $\mathcal{H}$ with the normalization $f(0)=0=f'(0)-1$. Also, let $\mathcal{S}=\{f\in\mathcal{A}:\ \ f \text{ is univalent in } \mathbb{D}\}$ and $\mathcal{S}^* := \mathcal{S}^*(0)\subset\mathcal{S}$ denote the class of all starlike (univalent) functions in $\mathbb{D}$. Here $\mathcal{S}^*(\beta)$ denotes the family of starlike functions of order β, i.e., functions $f\in\mathcal{S}$ such that [7]

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad z\in\mathbb{D},$$

where $0\leq\beta<1$. For $f\in\mathcal{H}$, the integral means

$$I_1(r,f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\theta}{|f(re^{i\theta})|^2}$$

and the estimates of I_1 are important in certain problems in fluid dynamics (see [8, 22, 23]). Recently the authors in [16] obtained that if $f\in\mathcal{S}^\star(\beta)$, then the estimate

$$L_1(r,f) := r^2 I_1(r,f) \leq \frac{\Gamma(5-4\beta)}{\Gamma^2(3-2\beta)}$$

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holds and the inequality is sharp. This settled the open problem of Gromova and Vasil'ev [8, p. 565]. Also, in the same paper the authors in [16] discussed the same problem for the class of α-spirallike functions of order β. Other than these recent results, nothing is known in the literature concerning the estimate for $L_1(r,f)$ for many geometric classes of functions from $\mathcal{S}$ or functions that are not necessarily univalent in $\mathbb{D}$. One of our aims is to state analogous results for many other situations.

Our second aim concerns Dirichlet integral. For $g\in\mathcal{H}$, we denote the area of the transform of image of $|z|<r$ under $w=g(z)$ by $\Delta(r,g)$, where $0<r\leq 1$. Thus for $g(z)=\sum_{n=0}^{\infty}b_nz^n$, we have

$$\Delta(r,g)=\int\int_{|z|<r}|g'(z)|^2\,dxdy=\pi\sum_{n=1}^{\infty}n|b_n|^2r^{2n}\qquad (z=x+iy). \tag{1}$$

We call g a Dirichlet-finite function whenever $\Delta(1,g)$, the area covered by the mapping $z\to g(z)$ for $|z|<1$, is finite. In [24], Yamashita discussed the extremal problems

$$A(r)=\max_{f\in\mathcal{F}}\Delta\left(r,\frac{z}{f(z)}\right)\quad\text{for } 0<r\leq 1,$$

where $\mathcal{F}$ represents certain subclasses of $\mathcal{S}$, and conjectured that $A(r)=\pi r^2$ whenever $\mathcal{F}$ consisting of normalized analytic univalent functions f such that $f(\mathbb{D})$ convex. In a recent article, the present authors consider such problems in [11] and settled the conjecture of Yamashita by proving a general version of this conjecture for the class $\mathcal{S}^{\star}(\beta)$ and later in [16] also for the class of α-spirallike functions of order β. In this paper, we investigate Yamashita's conjecture for some classes of functions including cases where functions are not necessarily univalent in $\mathbb{D}$.

The paper is organized as follows. In the remaining part of this section, we introduce some basic classes that have been studied by a number of researchers, and state main results and remarks on these results. The proofs of these results together with the proof of Lemma 1 below will be given in Section 2.

1.1. **The class $\mathcal{U}(\lambda)$.** For $\lambda>0$, let $\mathcal{U}(\lambda)$ denote the class of functions $f\in\mathcal{A}$ such that $|U_f(z)-1|<\lambda$ in $\mathbb{D}$, where

$$U_f(z):=\left(\frac{z}{f(z)}\right)^2f'(z). \tag{2}$$

Also, we let $\mathcal{U}:=\mathcal{U}(1)$. According to Aksentév's theorem [1] (see also [12]), the strict inclusion $\mathcal{U}\subsetneq\mathcal{S}$ holds and hence, for $0<\lambda\leq 1$, $\mathcal{U}(\lambda)\subset\mathcal{S}$. Set

$$\mathcal{U}_2(\lambda)=\{f\in\mathcal{U}(\lambda):\ f''(0)=0\}$$

so that $\mathcal{U}_2:=\mathcal{U}_2(1)$. It is known that $\mathcal{U}$ is not a subset of $\mathcal{S}^*$ as the function

$$f_1(z)=\frac{z}{1+\frac{1}{2}z+\frac{1}{2}z^3}$$

demonstrates. We observe that mappings $f \in \mathcal{S}$ can be associated with the mappings $F \in \Sigma$, namely, the class of univalent meromorphic functions F of the form,

$$F(\zeta) = \zeta + \sum_{n=0}^{\infty} c_n \zeta^{-n}, \quad |\zeta| > 1,$$

which satisfies the condition $F(\zeta) \neq 0$ for $|\zeta| > 1$, by the correspondence

$$F(\zeta) = \frac{1}{f(1/\zeta)}, \quad |\zeta| > 1.$$

Using the change of variable $\zeta = 1/z$, the association $f(z) = 1/F(1/z)$ quickly yields the formula

$$F'(\zeta) + 1 = -\left(\frac{z}{f(z)}\right)^2 f'(z) + 1,$$

so that, as a consequence of Schwarz's lemma, $f \in \mathcal{U}(\lambda)$ if and only if $|F'(\zeta) + 1| < \lambda|\zeta|^{-2}$ for $|\zeta| > 1$. Some facts about the class $\mathcal{U}$ may now be recalled. Each function in

$$\mathcal{S}_{\mathbb{Z}} = \left\{ z,\ \frac{z}{(1 \pm z)^2},\ \frac{z}{1 \pm z},\ \frac{z}{1 \pm z^2},\ \frac{z}{1 \pm z + z^2} \right\}$$

belongs to $\mathcal{U}$. Also, it is well-known that functions in $\mathcal{S}_{\mathbb{Z}}$ are the only functions in $\mathcal{S}$ having integral coefficients in the power series expansions of $f \in \mathcal{S}$ (see [6]). It is a simple exercise to see that $\mathcal{S}_{\mathbb{Z}} \subset \mathcal{S}^*$ and in particular, the Koebe function $k(z) = z/(1-z)^2$ belongs to both $\mathcal{U}$ and $\mathcal{S}^*$.

We begin our studies of $\mathcal{U}(\lambda)$ by a lemma on the coefficients of

$$\frac{z}{f(z)} \ = \ 1 + \sum_{k=1}^{\infty} b_k z^k,$$

where

$$f(z) \ = \ z + \sum_{k=1}^{\infty} a_k z^k \ \in \mathcal{U}(\lambda).$$

Clearly, $-b_1 = a_2 := a_2(f) = \frac{f''(0)}{2}$ and we use this notation throughout.

Lemma 1. *Let $f \in \mathcal{U}(\lambda)$ for some $0 < \lambda \leq 1$, and let $t \leq 2$. Then we have*

$$\sum_{k=2}^{\infty} k^t\, |b_k|^2 r^{2k} \ \leq \ 2^t\, \lambda^2 r^4.$$

If in Lemma 1 we take $t = 0$, then we get

$$r^2 I_1(r,f) \quad = \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{r^2}{|f(re^{i\theta})|^2}\, d\theta = 1 + \sum_{k=1}^{\infty} |b_k|^2 r^{2k} \leq 1 + |a_2|^2 r^2 + \lambda^2 r^4.$$

Equality occurs in the above inequalities if $f(z) = z/(1+bz+\lambda z^2)$, where $|b| \leq 1+\lambda$. This proves

Theorem 1. *Let $f \in \mathcal{U}(\lambda)$ for some $0 < \lambda \leq 1$. Then we have*

$$L_1(r,f) := r^2 I_1(r,f) \leq 1 + |a_2|^2 + \lambda^2.$$

In particular, $L_1(r,f) \leq 6$ for $f \in \mathcal{U}$, and $L_1(r,f) \leq 2$ for $f \in \mathcal{U}_2$. All the inequalities are sharp.

It is worth pointing out that the bound 6 works for both $\mathcal{U}$ and $\mathcal{S}^*$ although one is not contained in the other.

If we let $t = 1$ in Lemma 1, then we get

$$\begin{aligned} \pi^{-1}\Delta\left(r, \frac{z}{f(z)}\right) &= \sum_{n=1}^{\infty} n|b_n|^2 r^{2n} = |b_1|^2 r^2 + \sum_{n=2}^{\infty} n|b_n|^2 r^{2n} \\ &\leq |a_2|^2 r^2 + 2r^4\lambda^2. \end{aligned}$$

For the function $f_0 \in \mathcal{U}(\lambda)$ defined by $\frac{z}{f_0(z)} = 1 + bz + \lambda z^2$, we have the equality

$$\pi^{-1}\Delta\left(r, \frac{z}{f_0(z)}\right) = r^2\left(|b|^2 + 2r^2\lambda^2\right),$$

where $|b| \leq 1 + \lambda$. Clearly,

$$\max_{f \in \mathcal{U}_2(\lambda)} \Delta\left(r, \frac{z}{f(z)}\right) = 2\lambda^2 \pi r^4$$

and the bound is sharp for $f_0(z) = z/(1 - \lambda z^2)$ and its rotations. This proves

Theorem 2. *If $f \in \mathcal{U}(\lambda)$ for some $0 < \lambda \leq 1$, then*

$$\max_{f \in \mathcal{U}(\lambda)} \Delta\left(r, \frac{z}{f(z)}\right) = \pi r^2\left(|a_2|^2 + 2r^2\lambda^2\right).$$

In particular,

$$\max_{f \in \mathcal{U}} \Delta\left(r, \frac{z}{f(z)}\right) = 2\pi r^2\left(2 + r^2\right) \quad \text{and} \quad \max_{f \in \mathcal{U}_2(\lambda)} \Delta\left(r, \frac{z}{f(z)}\right) = 2\lambda^2\pi r^4.$$

The results are sharp.

It is interesting to observe that the conclusion for the class $\mathcal{U}$ was obtained in [11, Theorem 4] for the subclass $\mathcal{S}_+$ of functions $f \in \mathcal{S}$ having the form

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots,$$

where $b_n \geq 0$ for $n \geq 2$. Thus, Theorem 2 includes an analogous result of [11, Theorem 4] for the class $\mathcal{U}$.

1.2. **The class $\mathcal{R}(\alpha,\lambda)$.** For $f \in \mathcal{A}$, let

$$R_f(\alpha; z) = (1-\alpha)(f(z)/z) + \alpha f'(z) - 1, \quad z \in \mathbb{D},$$

where α is a complex constant. We say that a function $f \in \mathcal{A}$ is said to be in $\mathcal{R}(\alpha,\lambda)$ if $|R_f(\alpha;z)| < \lambda$ in $\mathbb{D}$, for some $\lambda > 0$. Further for convenience, we let $\mathcal{R}(1,\lambda) = \mathcal{R}(\lambda)$, and $\mathcal{R}(1) = \mathcal{R}$. These classes have been extensively studied in the literature. Functions in $\mathcal{R}(\lambda)$ are known to be univalent whenever $0 < \lambda \leq 1$ and functions in $\mathcal{R}(\lambda)$ for $\lambda > 2/\sqrt{5}$ are not necessarily belonging to $\mathcal{S}^*$. We now recall the following lemma which is indeed special cases of a general result from [14] (see also [13, 21]).

Lemma A. *The following assertions are valid.*

(a) *Each* $f \in \mathcal{R}(2/\sqrt{5})$ *belongs to* $\mathcal{S}^*$.

(b) *Each* $f \in \mathcal{R}(3/\sqrt{10})$ *belongs to* $\mathcal{S}^*$ *if* $f''(0) = 0$.

The numbers $2/\sqrt{5}$ and $3/\sqrt{10}$ in Lemma A were proved to be sharp (see for instance [5, 19]). We refer to [15] for many other interesting properties of the class $\mathcal{R}(\alpha, \lambda)$.

Theorem 3. *Let* $\alpha \in \mathbb{C}$ *and* $\lambda > 0$ *such that* $0 < \lambda < |1+\alpha|$. *If* $f \in \mathcal{R}(\alpha, \lambda)$, *then we have*

$$L_1(r, f) := r^2 I_1(r, f) \leq \frac{|1+\alpha|^2}{|1+\alpha|^2 - \lambda^2}.$$

The estimate is sharp for $f(z) = z + \frac{\lambda}{1+\alpha} z^2$. *In particular,* $L_1(r, f) \leq 4/(4-\lambda^2)$ *for* $f \in \mathcal{R}(\lambda)$ *with* $\lambda \in (0, 2)$. *As special cases, the following assertions are valid:*

(a) $L_1(r, f) \leq 4/3$ *for* $f \in \mathcal{R}$

(b) $L_1(r, f) \leq 5/4$ *for* $f \in \mathcal{R}(2/\sqrt{5})$.

All the inequalities are sharp.

Remark. Although $L_1(r, f) \leq 6$ for $f \in \mathcal{S}^*$, according to Theorems 1 and 3, there are univalent functions that are not necessarily starlike, as well as non-univalent functions, such that $L_1(r, f) \leq a$ with $a < 6$.

To find the analog of Theorem 2 seems not very easy, but among the possibilities to use the methods of the proof of Lemma 1, we prove the following

Theorem 4. *Let* $\alpha \in \mathbb{C}$ *and* $\lambda > 0$ *such that* $0 < \lambda < |1+\alpha|$ *and* $c = \frac{\lambda}{1+\alpha}$, $D = |c|$. *Let further*

$$g(z) = \int_0^z \frac{t}{f(t)}\, dt.$$

If $f \in \mathcal{R}(\alpha, \lambda)$, *then we have*

$$\Delta(r, g) \leq -\frac{\pi}{D^2} \log(1 - D^2 r^2).$$

The estimate is sharp for $f(z) = z + c\, z^2$.

1.3. **The class $\mathcal{S}(A, B)$.** Next, we consider

$$\mathcal{S}(A, B) = \left\{ f \in \mathcal{A} :\ f'(z) \prec \frac{1+Az}{1+Bz},\ z \in \mathbb{D} \right\},$$

where $-1 \leq B \leq 1$ and $A > B$. Here $\prec$ denotes the usual subordination. For $\beta < 1$, $\mathcal{S}(1-2\beta, -1) =: \mathcal{S}(\beta)$ denotes the usual normalized class of all functions f analytic and satisfies the condition $\operatorname{Re} f'(z) > \beta$ in $\mathbb{D}$. Functions in $\mathcal{S}(0)$ are known to be univalent in $\mathbb{D}$ (see [10]) and hence, functions in $\mathcal{S}(A, B)$ are included in the class $\mathcal{S}(0)$ whenever the condition $-1 \leq B < A \leq 1$ is satisfied. Note that

$\mathcal{S}(1,-1) := \mathcal{S}(0)$ and for $0 < A \leq 1$, the class $\mathcal{S}(A,0)$ coincides with the class $\mathcal{R}(A)$ defined previously. Thus, we need to deal with only the case $B \neq 0$.

Theorem 5. *Let $f \in \mathcal{S}(A,B)$ for some $-1 \leq B \leq 1$ with $B \neq 0$, and $A > B$ be such that*

$$q(z) = \frac{A}{B} + \left(1 - \frac{A}{B}\right)\frac{\log(1+Bz)}{Bz} \neq 0, \quad z \in \mathbb{D}. \tag{3}$$

Then we have

$$L_1(r,f) := r^2 I_1(r,f) \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\theta}{|q(re^{i\theta})|^2}.$$

The inequality is sharp.

Remark. It is not quite clear for which pairs (A,B) the condition (3) holds, but one can find sufficient conditions. We mention two of them. Since for $|z| = r$

$$\begin{aligned}
\mathrm{Re}\,(q(z)) &= \mathrm{Re}\left(1 + (B-A)\sum_{k=1}^{\infty}\frac{(-1)^k B^{k-1} z^k}{k+1}\right) \\
&\geq 1 - (A-B)\sum_{k=1}^{\infty}\frac{|B|^{k-1} r^k}{k+1} \\
&= \frac{A-B+|B|}{|B|} + \left(\frac{A-B}{|B|}\right)\frac{\log(1-|B|r)}{|B|r},
\end{aligned}$$

the condition (3) is satisfied, if the last term is bigger than zero for all $r \in [0,1)$.

The second case we want to consider is the class $\mathcal{S}(1-2\beta,-1) = \mathcal{S}(\beta)$, $\beta < 1$. In this case, we have

$$q(z) = 1 + 2(1-\beta)\sum_{k=1}^{\infty}\frac{z^k}{k+1} = (2\beta-1) - 2(1-\beta)\frac{\log(1-z)}{z},$$

and since (see [9])

$$\mathrm{Re}\left(-\frac{\log(1-z)}{z}\right) \geq \frac{\log(1+r)}{r} \quad \text{for } |z| = r,$$

it follows that

$$\mathrm{Re}\,(q(z)) \geq (2\beta-1) + 2(1-\beta)\frac{\log(1+r)}{r}.$$

Letting $r \to 1$, we find that $\mathrm{Re}\,(q(z)) > 0$ in $\mathbb{D}$ for

$$\beta \geq -\frac{1}{2}\left(\frac{2\log 2 - 1}{1 - \log 2}\right) \approx -0.63.$$

Hence the bound of Theorem 5 is valid for the functions in $\mathcal{S}(\beta)$ at least for these values of β.

Finally, we would like to mention that for $-1 \leq B < A \leq 1$ and in turn for $\beta \in [0,1)$ the non-vanishing condition (3) for q is obviously fulfilled from the mentioned reason.

2. Proofs of the Main Results

Proof of Lemma 1. Suppose that $f \in \mathcal{U}(\lambda)$. Then, by the power series representation of $z/f(z)$ and (2), each $f \in \mathcal{U}$ can be written as

$$-z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} - 1 = \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 = \sum_{k=2}^{\infty}(1-k)\, b_k z^k = \lambda\omega(z)$$

where $\omega \in \mathcal{B}_0$. Here $\mathcal{B}_0$ denotes the class of analytic functions $\omega(z)$ in $\mathbb{D}$ such that $\omega(0) = \omega'(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{D}$. We have $|\omega(z)| \leq |z|^2$ in $\mathbb{D}$ (by Schwarz's lemma). Taking H^2-norm inequality for $\omega(z)$, it follows that

$$\sum_{k=2}^{\infty}(k-1)^2|b_k|^2 r^{2k} \;\leq\; \lambda^2 r^4$$

from which we get that for each $n \geq 2$ the inequality

$$\sum_{k=2}^{n}(k-1)^2|b_k|^2 r^{2k} \;\leq\; \lambda^2 r^4$$

is valid. Now, we take these inequalities for $n = 2, \dots, N$, and multiply the N-th inequality by the factor

$$\frac{N^t}{(N-1)^2},$$

and, for $n = 2, \dots, N-1$, the n-th inequality by the factor

$$\frac{n^t}{(n-1)^2} - \frac{(n+1)^t}{n^2} \;>\; 0.$$

Adding up these modified inequalities results in the inequality

$$\sum_{n=2}^{N} n^t\, |b_n|^2 r^{2n} \;\leq\; 2^t\, \lambda^2 r^4.$$

If we let $N \to \infty$, we see that the proof of the lemma is complete. $\square$

The proofs of Theorems 3 and 5 rely on a special case of the following lemma due to Ruscheweyh and Stankiewicz [20].

Lemma B. *Suppose that $f, g \in \mathcal{H}$, and F, G are convex in $\mathbb{D}$. If $f \prec F$ and $g \prec G$, then $f * g \prec F * G$. Here $*$ denotes the usual Hadamard product/convolution between two analytic functions.*

Proof of Theorem 3. Let $f \in \mathcal{R}(\alpha, \lambda)$. Then, we may write

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec 1 + \lambda z, \quad z \in \mathbb{D} \tag{4}$$

and hence, we easily have

$$\frac{f(z)}{z} \prec 1 + \frac{\lambda}{1+\alpha} z, \quad z \in \mathbb{D}. \tag{5}$$

Indeed if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then we may write (4) as

$$1 + \sum_{n=2}^{\infty}(1 - \alpha + n\alpha)a_n z^{n-1} \prec 1 + \lambda z, \quad z \in \mathbb{D},$$

and since $\phi_\alpha(z) = 1 + \sum_{n=1}^{\infty} \frac{1/\alpha}{n+(1/\alpha)} z^n$ is convex in $\mathbb{D}$, it follows from [20] (see Lemma B) that

$$\frac{f(z)}{z} = \left(1 + \sum_{n=2}^{\infty}(1 - \alpha + n\alpha)a_n z^{n-1}\right) * \phi_\alpha(z) \prec (1 + \lambda z) * \phi_\alpha(z) = 1 + \frac{\lambda}{1+\alpha} z,$$

and (5) follows. Again, since $\Phi_{\alpha,\lambda}(z) = 1 + \frac{\lambda}{1+\alpha} z$ is univalent in $\mathbb{D}$ and, by the condition on α and λ, $\Phi_{\alpha,\lambda}$ is non-vanishing in $\mathbb{D}$, it follows that

$$\frac{z}{f(z)} \prec \frac{1}{1 + (\lambda/(1+\alpha))z} = 1 + \sum_{k=1}^{\infty} c_k z^k, \quad z \in \mathbb{D}. \tag{6}$$

As in the proof of Theorem 1, the last relation gives

$$1 + \sum_{k=1}^{\infty} |b_k|^2 r^{2k} \leq 1 + \sum_{k=1}^{\infty} |c_k|^2 r^{2k} = 1 + \sum_{k=1}^{\infty} \left|\frac{\lambda}{1+\alpha}\right|^{2k} r^{2k}$$

so that from the last estimates we obtain that

$$\begin{aligned} r^2 I_1(r,f) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r^2}{|f(re^{i\theta})|^2}\, d\theta = 1 + \sum_{k=1}^{\infty} |b_k|^2 r^{2k} \\ &\leq 1 + \sum_{k=1}^{\infty} |b_k|^2 \\ &\leq 1 + \sum_{k=1}^{\infty} \left|\frac{\lambda}{1+\alpha}\right|^{2k} = \frac{|1+\alpha|^2}{|1+\alpha|^2 - \lambda^2}. \end{aligned}$$

The result is sharp for $f(z) = z + \frac{\lambda}{1+\alpha} z^2$. □

Proof of Theorem 4. Let $f \in \mathcal{R}(\alpha, \lambda)$. Then (6) holds and we write it in the form

$$\frac{z}{f(z)} = \frac{1}{1 + cz\omega(z)},$$

where ω is analytic in $\mathbb{D}$ and $|\omega(z)| \leq 1$ for $z \in \mathbb{D}$. The resulting equation

$$\frac{z}{f(z)} - 1 = \frac{z^2}{f(z)}\, c\,\omega(z)$$

delivers, by Clunie's method (see [2], and also [3, 17, 18]), for $n \in \mathbb{N}$ the inequalities

$$\sum_{k=1}^{n-1} |b_k|^2 r^{2k}(1 - D^2 r^2) + |b_n|^2 r^{2n} \leq D^2 r^2. \tag{7}$$

Further, we calculate

$$\Delta(r,g) = \pi\left(r^2 + \sum_{k=1}^{\infty}\frac{|b_k|^2 r^{2k+2}}{k+1}\right)$$

Now, we take steps analogous to those in the proof of the preceding lemma. We take the inequalities (7) for $n = 1, \dots, N$, multiply the N-th equation by $\frac{1}{N+1}$, and for $n = 1, \dots, N-1$ by the factor

$$\sum_{j=0}^{N-n-1}\left(\frac{1}{n+1+j} - \frac{1}{n+2+j}\right)(D^2r^2)^j + \frac{(D^2r^2)^{N-n}}{N+1}.$$

If we add these modified inequalities and let $N \to \infty$, we get the assertion of the theorem □

Proof of Theorem 5. Let $f \in \mathcal{S}(A,B)$ for some $-1 \leq B \leq 1$ with $B \neq 0$, and $A > B$. Then we can write

$$\frac{f(z)}{z} * \frac{1}{(1-z)^2} \prec \frac{1+Az}{1+Bz}.$$

Taking convolution with the convex function $\phi(z) = 1 + \sum_{n=1}^{\infty}\frac{1}{n+1}z^n$ it follows from [20] (see Lemma B) that

$$\frac{f(z)}{z} \prec \frac{1+Az}{1+Bz} * \phi(z) = q(z),$$

where $q(z) = \sum_{k=0}^{\infty} d_k z^k$ is given by (3) and

$$d_k = \left(1 - \frac{A}{B}\right)\frac{(-1)^k B^k}{k+1} \quad \text{for } k \geq 2$$

and $d_0 = 1$. Then, by the condition on q, we see that $1/q$ is a well-defined analytic function in the unit disk $\mathbb{D}$. This observation shows that

$$\frac{z}{f(z)} \prec \frac{1}{q(z)}, \quad z \in \mathbb{D}.$$

By [16, Lemma A] we have, for each real p and $0 \leq r < 1$,

$$\text{(8)} \qquad \frac{r^2}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta}{|f(re^{i\theta})|^{2p}} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta}{|q(re^{i\theta})|^{2p}}.$$

Setting $p = 1$ in the last relation proves the assertion of the theorem. It is easily seen that $f(z) = zq(z) \in \mathcal{S}(A,B)$. Hence the inequality is sharp. □

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