INTEGRAL MEANS AND DIRICHLET INTEGRAL FOR ANALYTIC FUNCTIONS

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Abstract. For normalized analytic functions \( f \) in the unit disk, the estimate of the integral means

\[
L_1(r, f) := r^2 \int_{-\pi}^{\pi} \frac{d\theta}{|f(re^{i\theta})|^2}
\]

is important in certain problems in fluid dynamics, especially when the functions \( f(z) \) are non-vanishing in the punctured unit disk \( 0 < |z| < 1 \). We consider the problem of finding the extremal function \( f \) which maximizes the integral means \( L_1(r, f) \). In addition, for certain class \( \mathcal{F} \) of analytic functions, we solve the extremal problem for the Yamashita functional

\[
A(r) = \max_{f \in \mathcal{F}} \Delta \left( r, \frac{z}{f(z)} \right) \quad \text{for} \quad 0 < r \leq 1.
\]

1. Preliminaries and Main Results

Denote by \( \mathcal{H} \) the family of all functions \( f \) which are analytic in the unit disk \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) and by \( \mathcal{A} \) the subfamily of \( \mathcal{H} \) with the normalization \( f(0) = 0 = f'(0) - 1 \). Also, let \( \mathcal{S} = \{ f \in \mathcal{A} : \text{f is univalent in } \mathbb{D} \} \) and \( \mathcal{S}^* := \mathcal{S}^*(0) \subset \mathcal{S} \) denote the class of all starlike (univalent) functions in \( \mathbb{D} \). Here \( \mathcal{S}^*(\beta) \) denotes the family of starlike functions of order \( \beta \), i.e., functions \( f \in \mathcal{S} \) such that [7]

\[
\text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > \beta, \quad z \in \mathbb{D},
\]

where \( 0 \leq \beta < 1 \). For \( f \in \mathcal{H} \), the integral means

\[
I_1(r, f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{|f(re^{i\theta})|^2}
\]

and the estimates of \( I_1 \) are important in certain problems in fluid dynamics (see [8, 22, 23]). Recently the authors in [16] obtained that if \( f \in \mathcal{S}^*(\beta) \), then the estimate

\[
L_1(r, f) := r^2 I_1(r, f) \leq \frac{\Gamma(5 - 4\beta)}{\Gamma^2(3 - 2\beta)}
\]

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holds and the inequality is sharp. This settled the open problem of Gromova and Vasil’ev [8, p. 565]. Also, in the same paper the authors in [16] discussed the same problem for the class of \( \alpha \)-spirallike functions of order \( \beta \). Other than these recent results, nothing is known in the literature concerning the estimate for \( L_1(r, f) \) for many geometric classes of functions from \( S \) or functions that are not necessarily univalent in \( \mathbb{D} \). One of our aims is to state analogous results for many other situations.

Our second aim concerns Dirichlet integral. For \( g \in \mathcal{H} \), we denote the area of the transform of image of \( |z| < r \) under \( w = g(z) \) by \( \Delta(r, g) \), where \( 0 < r \leq 1 \). Thus for \( g(z) = \sum_{n=0}^{\infty} b_n z^n \), we have

\[
\Delta(r, g) = \int \int_{|z|<r} |g'(z)|^2 \, dx \, dy = \pi \sum_{n=1}^{\infty} n|b_n|^2 r^{2n} \quad (z = x + iy).
\]

We call \( g \) a Dirichlet-finite function whenever \( \Delta(1, g) \), the area covered by the mapping \( z \to g(z) \) for \( |z| < 1 \), is finite. In [24], Yamashita discussed the extremal problems

\[
A(r) = \max_{f \in \mathcal{F}} \Delta \left( r, \frac{z}{f(z)} \right) \quad \text{for} \ 0 < r \leq 1,
\]

where \( \mathcal{F} \) represents certain subclasses of \( S \), and conjectured that \( A(r) = \pi r^2 \) whenever \( \mathcal{F} \) consisting of normalized analytic univalent functions \( f \) such that \( f(\mathbb{D}) \) convex. In a recent article, the present authors consider such problems in [11] and settled the conjecture of Yamashita by proving a general version of this conjecture for the class \( S^*(\beta) \) and later in [16] also for the class of \( \alpha \)-spirallike functions of order \( \beta \).

In this paper, we investigate Yamashita’s conjecture for some classes of functions including cases where functions are not necessarily univalent in \( \mathbb{D} \).

The paper is organized as follows. In the remaining part of this section, we introduce some basic classes that have been studied by a number of researchers, and state main results and remarks on these results. The proofs of these results together with the proof of Lemma 1 below will be given in Section 2.

### 1.1. The class \( \mathcal{U}(\lambda) \)

For \( \lambda > 0 \), let \( \mathcal{U}(\lambda) \) denote the class of functions \( f \in \mathcal{A} \) such that \( |U_f(z) - 1| < \lambda \) in \( \mathbb{D} \), where

\[
U_f(z) := \left( \frac{z}{f(z)} \right)^2 f'(z).
\]

Also, we let \( \mathcal{U} := \mathcal{U}(1) \). According to Aksentév’s theorem [1] (see also [12]), the strict inclusion \( \mathcal{U} \subsetneq S \) holds and hence, for \( 0 < \lambda \leq 1 \), \( \mathcal{U}(\lambda) \subset S \). Set

\[
\mathcal{U}_2(\lambda) = \{ f \in \mathcal{U}(\lambda) : f''(0) = 0 \}
\]

so that \( \mathcal{U}_2 := \mathcal{U}_2(1) \). It is known that \( \mathcal{U} \) is not a subset of \( S^* \) as the function

\[
f_1(z) = \frac{z}{1 + \frac{1}{2}z + \frac{1}{3}z^3}
\]
Integral Means and Dirichlet-finiteness

We observe that mappings \( f \in \mathcal{S} \) can be associated with the mappings \( F \in \Sigma \), namely, the class of univalent meromorphic functions \( F \) of the form,

\[
F(\zeta) = \zeta + \sum_{n=0}^{\infty} c_n \zeta^{-n}, \quad |\zeta| > 1,
\]

which satisfies the condition \( F(\zeta) \neq 0 \) for \( |\zeta| > 1 \), by the correspondence \( F(\zeta) = \frac{1}{f(1/\zeta)} \), \( |\zeta| > 1 \).

Using the change of variable \( \zeta = 1/z \), the association \( f(z) = 1/F(1/z) \) quickly yields the formula

\[
F'(\zeta) + 1 = - \left( \frac{z}{f(z)} \right)^2 f'(z) + 1,
\]

so that, as a consequence of Schwarz’s lemma, \( f \in U(\lambda) \) if and only if \( |F'(\zeta) + 1| < \lambda |\zeta|^{-2} \) for \( |\zeta| > 1 \). Some facts about the class \( U \) may now be recalled. Each function in \( \mathcal{S}_Z = \{ z, z(1 \pm z), z^2, z^2(1 \pm z), z^2 + z^2 \} \) belongs to \( U \). Also, it is well-known that functions in \( \mathcal{S}_Z \) are the only functions in \( \mathcal{S} \) having integral coefficients in the power series expansions of \( f \in \mathcal{S} \) (see [6]).

We begin our studies of \( U(\lambda) \) by a lemma on the coefficients of

\[
\frac{z}{f(z)} = 1 + \sum_{k=1}^{\infty} b_k z^k,
\]

where

\[
f(z) = z + \sum_{k=1}^{\infty} a_k z^k \in U(\lambda).
\]

Clearly, \(-b_1 = a_2 := a_2(f) = \frac{f''(0)}{2}\) and we use this notation throughout.

**Lemma 1.** Let \( f \in U(\lambda) \) for some \( 0 < \lambda \leq 1 \), and let \( t \leq 2 \). Then we have

\[
\sum_{k=2}^{\infty} k^t |b_k|^2 r^{2k} \leq 2^t \lambda^2 r^4.
\]

If in Lemma 1 we take \( t = 0 \), then we get

\[
r^2 I_1(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{r^2}{|f(re^{i\theta})|^2} d\theta = 1 + \sum_{k=1}^{\infty} |b_k|^2 r^{2k} \leq 1 + |a_2|^2 r^2 + \lambda^2 r^4.
\]

Equality occurs in the above inequalities if \( f(z) = z/(1+bz+\lambda z^2) \), where \(|b| \leq 1+\lambda\). This proves
**Theorem 1.** Let $f \in \mathcal{U}(\lambda)$ for some $0 < \lambda \leq 1$. Then we have

$$L_1(r, f) := r^2 I_1(r, f) \leq 1 + |a_2|^2 + \lambda^2.$$ 

In particular, $L_1(r, f) \leq 6$ for $f \in \mathcal{U}$, and $L_1(r, f) \leq 2$ for $f \in \mathcal{U}_2$. All the inequalities are sharp.

It is worth pointing out that the bound 6 works for both $\mathcal{U}$ and $\mathcal{S}^*$ although one is not contained in the other.

If we let $t = 1$ in Lemma 1, then we get

$$\pi^{-1} \Delta \left( r, \frac{z}{f(z)} \right) = \sum_{n=1}^{\infty} n|b_n|^2 r^{2n} = |b_1|^2 r^2 + \sum_{n=2}^{\infty} n|b_n|^2 r^{2n} \leq |a_2|^2 r^2 + 2r^4 \lambda^2.$$ 

For the function $f_0 \in \mathcal{U}(\lambda)$ defined by $z f_0(z) = 1 + bz + \lambda z^2$, we have the equality

$$\pi^{-1} \Delta \left( r, \frac{z}{f_0(z)} \right) = r^2 \left( |b|^2 + 2r^2 \lambda^2 \right),$$ 

where $|b| \leq 1 + \lambda$. Clearly, 

$$\max_{f \in \mathcal{U}(\lambda)} \Delta \left( r, \frac{z}{f(z)} \right) = 2 \lambda^2 \pi r^4$$

and the bound is sharp for $f_0(z) = z/(1 - \lambda z^2)$ and its rotations. This proves

**Theorem 2.** If $f \in \mathcal{U}(\lambda)$ for some $0 < \lambda \leq 1$, then

$$\max_{f \in \mathcal{U}(\lambda)} \Delta \left( r, \frac{z}{f(z)} \right) = \pi r^2 \left( |a_2|^2 + 2r^2 \lambda^2 \right).$$

In particular,

$$\max_{f \in \mathcal{U}} \Delta \left( r, \frac{z}{f(z)} \right) = 2 \pi r^2 (2 + r^2) \quad \text{and} \quad \max_{f \in \mathcal{U}(\lambda)} \Delta \left( r, \frac{z}{f(z)} \right) = 2 \lambda^2 \pi r^4.$$ 

The results are sharp.

It is interesting to observe that the conclusion for the class $\mathcal{U}$ was obtained in [11, Theorem 4] for the subclass $\mathcal{S}_+^*$ of functions $f \in \mathcal{S}$ having the form

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots,$$

where $b_n \geq 0$ for $n \geq 2$. Thus, Theorem 2 includes an analogous result of [11, Theorem 4] for the class $\mathcal{U}$.

1.2. **The class $\mathcal{R}(\alpha, \lambda)$.** For $f \in \mathcal{A}$, let 

$$R_f(\alpha; z) = (1 - \alpha)(f(z)/z) + \alpha f'(z) - 1, \quad z \in \mathbb{D},$$

where $\alpha$ is a complex constant. We say that a function $f \in \mathcal{A}$ is said to be in $\mathcal{R}(\alpha, \lambda)$ if $|R_f(\alpha; z)| < \lambda$ in $\mathbb{D}$, for some $\lambda > 0$. Further for convenience, we let $\mathcal{R}(1, \lambda) = \mathcal{R}(\lambda)$, and $\mathcal{R}(1) = \mathcal{R}$. These classes have been extensively studied in the literature. Functions in $\mathcal{R}(\lambda)$ are known to be univalent whenever $0 < \lambda \leq 1$ and functions in $\mathcal{R}(\lambda)$ for $\lambda > 2/\sqrt{3}$ are not necessarily belonging to $\mathcal{S}^*$. We now recall
the following lemma which is indeed special cases of a general result from [14] (see also [13, 21]).

**Lemma A.** The following assertions are valid.

(a) Each \( f \in \mathcal{R}(2/\sqrt{5}) \) belongs to \( S^* \).

(b) Each \( f \in \mathcal{R}(3/\sqrt{10}) \) belongs to \( S^* \) if \( f''(0) = 0 \).

The numbers \( 2/\sqrt{5} \) and \( 3/\sqrt{10} \) in Lemma A were proved to be sharp (see for instance [5, 19]). We refer to [15] for many other interesting properties of the class \( \mathcal{R}(\alpha, \lambda) \).

**Theorem 3.** Let \( \alpha \in \mathbb{C} \) and \( \lambda > 0 \) such that \( 0 < \lambda < |1 + \alpha| \). If \( f \in \mathcal{R}(\alpha, \lambda) \), then we have

\[
L_1(r, f) := r^2 I_1(r, f) \leq \frac{|1 + \alpha|^2}{|1 + \alpha|^2 - \lambda^2}.
\]

The estimate is sharp for \( f(z) = z + \frac{\lambda}{1+\alpha} z^2 \). In particular, \( L_1(r, f) \leq 4/(4 - \lambda^2) \) for \( f \in \mathcal{R}(\lambda) \) with \( \lambda \in (0, 2) \). As special cases, the following assertions are valid:

(a) \( L_1(r, f) \leq 4/3 \) for \( f \in \mathcal{R}(2/\sqrt{5}) \).

(b) \( L_1(r, f) \leq 5/4 \) for \( f \in \mathcal{R}(3/\sqrt{10}) \).

All the inequalities are sharp.

**Remark.** Although \( L_1(r, f) \leq 6 \) for \( f \in S^* \), according to Theorems 1 and 3, there are univalent functions that are not necessarily starlike, as well as non-univalent functions, such that \( L_1(r, f) \leq a \) with \( a < 6 \).

To find the analog of Theorem 2 seems not very easy, but among the possibilities to use the methods of the proof of Lemma 1, we prove the following

**Theorem 4.** Let \( \alpha \in \mathbb{C} \) and \( \lambda > 0 \) such that \( 0 < \lambda < |1 + \alpha| \) and \( c = \frac{\lambda}{1+\alpha} \), \( D = |c| \).

Let further

\[
g(z) = \int_0^z \frac{t}{f(t)} \, dt.
\]

If \( f \in \mathcal{R}(\alpha, \lambda) \), then we have

\[
\Delta(r, g) \leq -\frac{\pi}{D^2} \log(1 - D^2 r^2).
\]

The estimate is sharp for \( f(z) = z + cz^2 \).

1.3. The class \( S(A, B) \). Next, we consider

\[
S(A, B) = \left\{ f \in \mathcal{A} : f'(z) \prec \frac{1 + A z}{1 + B z}, \, z \in \mathbb{D} \right\},
\]

where \(-1 \leq B \leq 1\) and \( A > B \). Here \( \prec \) denotes the usual subordination. For \( \beta < 1 \), \( S(1 - 2\beta, -1) =: S(\beta) \) denotes the usual normalized class of all functions \( f \) analytic and satisfies the condition \( \text{Re} f'(z) > \beta \) in \( \mathbb{D} \). Functions in \( S(0) \) are known to be univalent in \( \mathbb{D} \) (see [10]) and hence, functions in \( S(A, B) \) are included in the class \( S(0) \) whenever the condition \(-1 \leq B < A \leq 1\) is satisfied. Note that
$S(1, -1) := S(0)$ and for $0 < A \leq 1$, the class $S(A, 0)$ coincides with the class $R(A)$ defined previously. Thus, we need to deal with only the case $B \neq 0$.

**Theorem 5.** Let $f \in S(A, B)$ for some $-1 \leq B \leq 1$ with $B \neq 0$, and $A > B$ be such that

\[
q(z) = \frac{A}{B} + \left(1 - \frac{A}{B}\right) \frac{\log(1 + Bz)}{Bz} \neq 0, \quad z \in \mathbb{D}.
\]

Then we have

\[
L_1(r, f) := r^2 I_1(r, f) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{|q(re^{i\theta})|^2}.
\]

The inequality is sharp.

**Remark.** It is not quite clear for which pairs $(A, B)$ the condition (3) holds, but one can find sufficient conditions. We mention two of them. Since for $|z| = r$

\[
\text{Re} \left( \frac{q(z)}{z} \right) = \text{Re} \left( 1 + (B - A) \sum_{k=1}^{\infty} \frac{(-1)^k B^{k-1} z^k}{k+1} \right)
\]

\[
\geq 1 - (A - B) \sum_{k=1}^{\infty} \frac{|B|^{k-1} r^k}{k+1}
\]

\[
= \frac{A - B + |B|}{|B|} + \left( \frac{A - B}{|B|} \right) \frac{\log(1 - |B|r)}{|B|r},
\]

the condition (3) is satisfied, if the last term is bigger than zero for all $r \in [0, 1)$.

The second case we want to consider is the class $S(1 - 2\beta, -1) = S(\beta)$, $\beta < 1$.

In this case, we have

\[
q(z) = 1 + 2(1 - \beta) \sum_{k=1}^{\infty} \frac{z^k}{k+1} = (2\beta - 1) - 2(1 - \beta) \frac{\log(1 - z)}{z},
\]

and since (see [9])

\[
\text{Re} \left( -\frac{\log(1 - z)}{z} \right) \geq \frac{\log(1 + r)}{r} \quad \text{for } |z| = r,
\]

it follows that

\[
\text{Re} \left( q(z) \right) \geq (2\beta - 1) + 2(1 - \beta) \frac{\log(1 + r)}{r}.
\]

Letting $r \to 1$, we find that $\text{Re} \left( q(z) \right) > 0$ in $\mathbb{D}$ for

\[
\beta \geq -\frac{1}{2} \left( \frac{2 \log 2 - 1}{1 - \log 2} \right) \approx -0.63.
\]

Hence the bound of Theorem 5 is valid for the functions in $S(\beta)$ at least for these values of $\beta$.

Finally, we would like to mention that for $-1 \leq B < A \leq 1$ and in turn for $\beta \in [0, 1)$ the non-vanishing condition (3) for $q$ is obviously fulfilled from the mentioned reason.
2. PROOFS OF THE MAIN RESULTS

Proof of Lemma 1. Suppose that \( f \in U(\lambda) \). Then, by the power series representation of \( z/f(z) \) and (2), each \( f \in U \) can be written as

\[
-z \left( \frac{z}{f(z)} \right)' + \frac{z}{f(z)} - 1 = \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 = \sum_{k=2}^{\infty} (1 - k) b_k z^k = \lambda \omega(z)
\]

where \( \omega \in B_0 \). Here \( B_0 \) denotes the class of analytic functions \( \omega(z) \) in \( D \) such that \( \omega(0) = \omega'(0) = 0 \) and \( |\omega(z)| < 1 \) for \( z \in D \). We have \( |\omega(z)| \leq |z|^2 \) in \( D \) (by Schwarz’s lemma). Taking \( H^2 \)-norm inequality for \( \omega(z) \), it follows that

\[
\sum_{k=2}^{\infty} (k - 1)^2 |b_k|^2 r^{2k} \leq \lambda^2 r^4
\]

from which we get that for each \( n \geq 2 \) the inequality

\[
\sum_{k=2}^{n} (k - 1)^2 |b_k|^2 r^{2k} \leq \lambda^2 r^4
\]

is valid. Now, we take these inequalities for \( n = 2, \ldots, N \), and multiply the \( N \)-th inequality by the factor

\[
\frac{N^t}{(N - 1)^2}
\]

and, for \( n = 2, \ldots, N - 1 \), the \( n \)-th inequality by the factor

\[
\frac{n^t}{(n - 1)^2} - \frac{(n + 1)^t}{n^2} > 0.
\]

Adding up these modified inequalities results in the inequality

\[
\sum_{n=2}^{N} n^t |b_n|^2 r^{2n} \leq 2^t \lambda^2 r^4.
\]

If we let \( N \to \infty \), we see that the proof of the lemma is complete. \( \square \)

The proofs of Theorems 3 and 5 rely on a special case of the following lemma due to Ruscheweyh and Stankiewicz [20].

Lemma B. Suppose that \( f, g \in H \), and \( F, G \) are convex in \( D \). If \( f \prec F \) and \( g \prec G \), then \( f \ast g \prec F \ast G \). Here \( \ast \) denotes the usual Hadamard product/convolution between two analytic functions.

Proof of Theorem 3. Let \( f \in R(\alpha, \lambda) \). Then, we may write

\[
(1 - \alpha) \frac{f(z)}{z} + \alpha f''(z) < 1 + \lambda z, \quad z \in D
\]

and hence, we easily have

\[
\frac{f(z)}{z} < 1 + \frac{\lambda}{1 + \alpha} z, \quad z \in D.
\]
Indeed if \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \), then we may write (4) as
\[
1 + \sum_{n=2}^{\infty} (1 - \alpha + n\alpha) a_n z^{n-1} < 1 + \lambda z, \quad z \in \mathbb{D},
\]
and since \( \phi_\alpha(z) = 1 + \sum_{n=1}^{\infty} \frac{1}{\pi + (1/\alpha)} z^n \) is convex in \( \mathbb{D} \), it follows from \([20]\) (see Lemma B) that
\[
\frac{f(z)}{z} = \left( 1 + \sum_{n=2}^{\infty} (1 - \alpha + n\alpha) a_n z^{n-1} \right) \cdot \phi_\alpha(z) \prec (1 + \lambda z) \prec \phi_\alpha(z) = 1 + \frac{\lambda}{1 + \alpha} z,
\]
and (5) follows. Again, since \( \Phi_{\alpha,\lambda}(z) = 1 + \frac{\lambda}{1 + \alpha} z \) is univalent in \( \mathbb{D} \) and, by the condition on \( \alpha \) and \( \lambda \), \( \Phi_{\alpha,\lambda} \) is non-vanishing in \( \mathbb{D} \), it follows that
\[
\frac{z}{f(z)} < \frac{1}{1 + (\lambda/(1 + \alpha))z} = 1 + \sum_{k=1}^{\infty} c_k z^k, \quad z \in \mathbb{D}.
\]
As in the proof of Theorem 1, the last relation gives
\[
1 + \sum_{k=1}^{\infty} |b_k|^2 r^{2k} \leq 1 + \sum_{k=1}^{\infty} |c_k|^2 r^{2k} = 1 + \sum_{k=1}^{\infty} \left| \frac{\lambda}{1 + \alpha} \right|^{2k} r^{2k}
\]
so that from the last estimates we obtain that
\[
r^2 I_1(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{r^2}{|f(re^{i\theta})|^2} d\theta = 1 + \sum_{k=1}^{\infty} |b_k|^2 r^{2k}
\]
\[
\leq 1 + \sum_{k=1}^{\infty} \left| b_k \right|^2
\]
\[
\leq 1 + \sum_{k=1}^{\infty} \left| \frac{\lambda}{1 + \alpha} \right|^{2k} = \frac{|1 + \alpha|^2}{|1 + \alpha|^2 - \lambda^2}.
\]
The result is sharp for \( f(z) = z + \frac{\lambda}{1 + \alpha} z^2 \).

**Proof of Theorem 4.** Let \( f \in \mathcal{R}(\alpha, \lambda) \). Then (6) holds and we write it in the form
\[
\frac{z}{f(z)} = \frac{1}{1 + cz\omega(z)},
\]
where \( \omega \) is analytic in \( \mathbb{D} \) and \( |\omega(z)| \leq 1 \) for \( z \in \mathbb{D} \). The resulting equation
\[
\frac{z}{f(z)} - 1 = \frac{z^2}{f(z)} c \omega(z)
\]
delivers, by Clunie’s method (see [2], and also [3, 17, 18]), for \( n \in \mathbb{N} \) the inequalities
\[
\sum_{k=1}^{n-1} |b_k|^2 r^{2k} (1 - D^2 r^2) + |b_n|^2 r^{2n} \leq D^2 r^2.
\]
Further, we calculate

$$\Delta(r, g) = \pi \left( r^2 + \sum_{k=1}^{\infty} \frac{|b_k|^2 r^{2k+2}}{k+1} \right)$$

Now, we take steps analogous to those in the proof of the preceding lemma. We take the inequalities (7) for $n = 1, \ldots, N$, multiply the $N$-th equation by $\frac{1}{N+1}$, and for $n = 1, \ldots, N - 1$ by the factor

$$\sum_{j=0}^{N-n-1} \left( \frac{1}{n+1+j} - \frac{1}{n+2+j} \right) (D^2 r^2)^j + \frac{(D^2 r^2)^N}{N+1}.$$ 

If we add these modified inequalities and let $N \to \infty$, we get the assertion of the theorem.

\[\square\]

**Proof of Theorem 5.** Let $f \in S(A, B)$ for some $-1 \leq B \leq 1$ with $B \neq 0$, and $A > B$. Then we can write

$$\frac{f(z)}{z} * \frac{1}{(1-z)^2} \preceq \frac{1 + Az}{1 + Bz}.$$ 

Taking convolution with the convex function $\phi(z) = 1 + \sum_{n=1}^{\infty} \frac{1}{n+1} z^n$ it follows from [20] (see Lemma B) that

$$\frac{f(z)}{z} \preceq \frac{1 + Az}{1 + Bz} * \phi(z) = q(z),$$

where $q(z) = \sum_{k=0}^{\infty} d_k z^k$ is given by (3) and

$$d_k = \left( 1 - \frac{A}{B} \right) \frac{(-1)^k B^k}{k+1} \text{ for } k \geq 2$$

and $d_0 = 1$. Then, by the condition on $q$, we see that $1/q$ is a well-defined analytic function in the unit disk $\mathbb{D}$. This observation shows that

$$\frac{z}{f(z)} \preceq \frac{1}{q(z)}, \quad z \in \mathbb{D}.$$ 

By [16, Lemma A] we have, for each real $p$ and $0 \leq r < 1$,

$$\frac{r^2}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{|f(re^{i\theta})|^2} \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{|q(re^{i\theta})|^2}.$$ 

Setting $p = 1$ in the last relation proves the assertion of the theorem. It is easily seen that $f(z) = zq(z) \in S(A, B)$. Hence the inequality is sharp.

\[\square\]

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