On elementary bounds for $\sum_{k=n}^{\infty} k^{-s}$

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Abstract

By means of elementary arguments we derive lower and upper bounds for $\sum_{k=n}^{\infty} k^{-s}$.

It is a direct consequence of the Cauchy integral criterion for series that

$$\sum_{k=n}^{\infty} k^{-s} \leq \int_{n}^{\infty} (t-1)^{-s} \, dt = \frac{(n-1)^{1-s}}{s-1}, \quad (1)$$

holds true for all $n \geq 2, s > 1$ (cf. [2]). This is a standard argument which is used in several text books on analytic number theory for instance in connection with the Riemann zeta-function (cf. [1]). The authors of the present note, who are concerned with lectures on number theory for undergraduates, discussed the problem of how to derive an upper bound for $\sum_{k=n}^{\infty} k^{-s}$ better than the one given in (1) by using only elementary arguments. The first result of these discussions is the following

**Theorem 1** Let $s \in \mathbb{R}, s > 1$. Then the inequality

$$\sum_{k=n}^{\infty} k^{-s} < \frac{(n - \frac{1}{2})^{1-s}}{s-1} \quad (2)$$

holds true for all $n \in \mathbb{N}$.

The proof of this theorem is based on the following

**Lemma 1** Let $s \in \mathbb{R}, s > 1$. Then the function

$$h_s(x) = (1-x)^{1-s} - (1+x)^{1-s} - 2(s-1)x \quad (3)$$

is strictly positive for all $x \in (0,1)$.

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Proof. Because of $h_s(0) = 0$ it remains to show that

$$h'_s(x) = (s - 1)[(1 - x)^{-s} + (1 + x)^{-s} - 2] > 0$$

is valid for all $x \in (0, 1)$. Again, $h'_s(0) = 0$ and we prove that

$$h''_s(x) = s(s - 1)[(1 - x)^{-s-1} - (1 + x)^{-s-1}]$$

is strictly positive in $(0, 1)$, that means,

$$(1 - x)^{-s-1} > (1 + x)^{-s-1}$$

for all $x \in (0, 1)$. But this is quite obvious. □

In order to prove Theorem 1 we show

$$k^{-s} < \frac{(k - \frac{1}{2})^{1-s}}{s-1} - \frac{(k + \frac{1}{2})^{1-s}}{s-1}$$

(4)

for all $k \in \mathbb{N}$. Thus, let $k \in \mathbb{N}$. Then $x := \frac{1}{2k} \in (0, 1)$ and Lemma 1 provides

$$(1 - \frac{1}{2k})^{1-s} - (1 + \frac{1}{2k})^{1-s} > 2(s - 1) \frac{1}{2k}.$$  

(5)

Multiplying both sides of (5) with $\frac{k^{1-s}}{s-1}$ yields (4). Now, we are done, since (4) implies

$$\sum_{k=n}^{\infty} k^{-s} < \sum_{k=n}^{\infty} \left( \frac{(k - \frac{1}{2})^{1-s}}{s-1} - \frac{(k + \frac{1}{2})^{1-s}}{s-1} \right)$$

and the right hand telescoping series is equal to $\frac{(n - \frac{1}{2})^{1-s}}{s-1}$.

Arguments, similar to those we used above, even show that the upper bound for $\sum_{k=n}^{\infty} k^{-s}$ given in Theorem 1 is strict in the following sense.

Theorem 2 Let $s \in \mathbb{R}, s > 1$. For any $c \in [0, \frac{1}{2})$ the inequality

$$\frac{(n-c)^{1-s}}{s-1} < \sum_{k=n}^{\infty} k^{-s}$$

holds true for all $n \in \mathbb{N}$ up to a finite number of possible exceptions.

The proof of this theorem is based on
Lemma 2 Let $s \in \mathbb{R}, s > 1$ and $c \in [0, \frac{1}{2})$. Then there exists an $\varepsilon > 0$ such that the function

$$h_{s,c} = (1 - cx)^{1-s} - (1 + (1 - c)x)^{1-s} - (s - 1)x$$

is strictly negative for all $x \in (0, \varepsilon)$.

Proof. We first observe that $h_{s,c}(0) = h'_{s,c}(0) = 0$ and $h''_{s,c}(0) < 0$ where

$$h'_{s,c}(x) = (s - 1)[c(1 - cx)^{-s} + (1 - c)(1 + (1 - c)x)^{-s}) - 1],$$

$$h''_{s,c}(x) = s(s - 1)[c^2(1 - cx)^{-s-1} - (1 - c)^2(1 + (1 - c)x)^{-s-1}],$$

$$h''_{s,c}(0) = s(s - 1)(c^2 - (1 - c)^2).$$

Thus, there exists an $\varepsilon > 0$ such that $h''_{s,c}(x)$ is strictly negative in $(0, \varepsilon)$ which completes the proof. □

We prove Theorem 2 and consider $k \in \mathbb{N}$ such that $k > \frac{1}{\varepsilon}$. Then Lemma 2 provides

$$(1 - \frac{c}{k})^{1-s} - (1 + \frac{1 - c}{k})^{1-s} - (s - 1)\frac{1}{k} < 0$$

which means

$$\frac{(k - c)^{1-s}}{s - 1} - \frac{(k + 1 - c)^{1-s}}{s - 1} < k^{-s}.$$  

We proceed as in the proof of Theorem 1 and obtain (6) for all $n > \frac{1}{\varepsilon}$.

The number of exceptions mentioned in Theorem 2 may be large, especially if $c$ is close to $\frac{1}{2}$. To see this, let $c_i \in [0, \frac{1}{2})$ for all $i \in \mathbb{N}$ and $n_i \in \mathbb{N}$ minimal such that

$$\frac{(n - c_i)^{1-s}}{s - 1} < \sum_{k=n}^{\infty} k^{-s} < \frac{(n - \frac{1}{2})^{1-s}}{s - 1}$$

holds true for all $n \geq n_i$. If $\lim_{i \to \infty} c_i = \frac{1}{2}$ then $\sup n_i = \infty$ since otherwise there exists an $n \in \mathbb{N}$ satisfying $n_i \leq n$ for all $i \in \mathbb{N}$, which yields the contradiction

$$\frac{(n - \frac{1}{2})^{1-s}}{s - 1} = \lim_{i \to \infty} \frac{(n - c_i)^{1-s}}{s - 1} \leq \sum_{k=n}^{\infty} k^{-s} < \frac{(n - \frac{1}{2})^{1-s}}{s - 1}.$$
References


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