COEFFICIENT CHARACTERIZATIONS AND SECTIONS FOR SOME UNIVALENT FUNCTIONS

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Abstract. Let $\mathcal{G}(\alpha)$ denote the class of locally univalent normalized analytic functions f in the unit disk $|z| < 1$ satisfying the condition

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{\alpha}{2} \quad \text{for } |z| < 1$$

and for some $0 < \alpha \leq 1$. In this paper, we first prove sharp coefficient bounds for the moduli of the Taylor coefficients a_n of $f \in \mathcal{G}(\alpha)$. Secondly, we determine the sharp bound for the Fekete-Szegö functional for functions in $\mathcal{G}(\alpha)$ with complex parameter λ. Thirdly, we present a convolution characterization for functions f belonging to $\mathcal{G}(\alpha)$ and as a consequence we obtain a number of sufficient coefficient conditions for f to belong to $\mathcal{G}(\alpha)$. Finally, we discuss the close-to-convexity and starlikeness of partial sums of $f \in \mathcal{G}(\alpha)$. In particular, each partial sum $s_n(z)$ of $f \in \mathcal{G}(1)$ is starlike in the disk $|z| \leq 1/2$ for $n \geq 11$. Moreover, for $f \in \mathcal{G}(1)$, we also have $\mathrm{Re}\,(s_n'(z)) > 0$ in $|z| \leq 1/2$ for $n \geq 11$.

1. Introduction

For $r > 0$, let $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ and $\mathbb{D} := \mathbb{D}_1$ be the open unit disk in the complex plane $\mathbb{C}$. Let $\mathcal{A}$ denote the family of all functions f that are analytic in $\mathbb{D}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

The class of univalent functions in $\mathcal{A}$ is traditionally denoted by $\mathcal{S}$. A function $f \in \mathcal{S}$ is called starlike if $f(\mathbb{D})$ is starlike (with respect to the origin), i.e. every line segment joining the origin to $w \in f(\mathbb{D})$ lies completely inside $f(\mathbb{D})$. The class of all starlike functions is denoted by $\mathcal{S}^*$, and functions $f \in \mathcal{S}^*$ are characterized by the condition

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}.$$

With the class $\mathcal{S}$ being of the first priority, its subclasses such as the class of convex and the class of close-to-convex mappings, respectively denoted by $\mathcal{C}$ and $\mathcal{K}$, have

2000 *Mathematics Subject Classification.* Primary: 30C45; Secondary: 30C20, 30C75, 30C80.

Key words and phrases. Analytic, univalent, starlike, close-to-convex and convex functions, coefficient inequality, area theorem, radius of univalency, subordination, convolution and the Fekete-Szegö functional.

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File: Ob-S1(12)˙final.tex, printed: 13-6-2012, 15.47.

The work of the first author was supported by MNZZS Grant, No. ON174017, Serbia.

been extensively studied for many years. These classes together with the class of typically-real analytic functions and the class functions convex in a direction are well understood and are studied in detail in the literature (see the books of Duren [?] and Goodman [?]).

If $f \in \mathcal{S}$ of the form (??) is arbitrary, then from the argument principle we see that the n-th *partial sum*/section $s_n(z) := z + \sum_{k=2}^{n} a_k z^k$ of f is univalent in each fixed compact subdisk $\overline{\mathbb{D}_r}$ of $\mathbb{D}$ provided that n is sufficiently large. In this way one can get univalent polynomials in $\mathcal{S}$ by setting $p_n(z) = r^{-1}s_n(rz)$. Consequently, the set of all univalent polynomials is dense with respect to the topology of locally uniformly convergence in the class $\mathcal{S}$. In 1928, Szegö [?] proved that if $f \in \mathcal{S}$ then each *partial sum* $s_n(z)$ of f is univalent in the disk $\mathbb{D}_{1/4}$. Clearly the partial sums of $f \in \mathcal{S}$ are not necessarily univalent throughout the unit disk $\mathbb{D}$ as the convex univalent function $f(z) = z/(1-z)$ demonstrates. Moreover, the second partial sum $s_2(z) = z + 2z^2$ of the Koebe function $k(z) = z/(1-z)^2$ is univalent in $\mathbb{D}_{1/4}$, and the radius $1/4$ is best possible. We refer to [?, §8.2, p. 241–246] and the survey article of Iliev [?] for some related investigations. The radius of starlikeness of the partial sum $s_n(z)$ of $f \in \mathcal{S}^*$ was proved by Robertson [?].

Theorem A. [?] (see also [?, Theorem 2, p. 1193]) *If $f \in \mathcal{S}$ is either starlike, or convex, or typically-real, or convex in the direction of imaginary axis, then there is an N such that, for $n \geq N$, the partial sum $s_n(z)$ has the same property in $\mathbb{D}_r$, where $r \geq 1 - 3n^{-1}\log n$.*

Later in [?] Ruscheweyh proved a stronger result by showing that the partial sums $s_n(z)$ of f are indeed starlike in $\mathbb{D}_{1/4}$ for functions f belonging not only to $\mathcal{S}$ but also to the closed convex hull of $\mathcal{S}$. Robertson [?] further showed that sections of the Koebe function $k(z)$ are univalent in the disk $|z| < 1 - 3n^{-1}\log n$ for $n \geq 5$, and that the constant 3 cannot be replaced by a smaller constant. In [?] Jenkins observed that a modification of the proof of Szegö [?] gives

$$r_n \geq 1 - (4+\epsilon)n^{-1}\log n$$

for each $\epsilon > 0$, and for all large n, so that $s_n(z)$ of $f \in \mathcal{S}$ is univalent in $|z| < r_n$. The Koebe function is extremal for many basic functionals on $\mathcal{S}$ and so it is of special interest. However, Bshouty and Hengartner [?, p. 408] pointed out that the Koebe function is no longer extremal for the radius of univalency of the partial sums of $f \in \mathcal{S}$. The exact (largest) radius of univalence r_n of $s_n(z)$ ($f \in \mathcal{S}$) remains an open problem (see [?, §8.2, p. 246] and [?, §6.4]). However, a well-known deeper result due to Ruscheweyh and Sheil-Small [?] (see also [?]) on convolution implies that if f belongs to $\mathcal{C}$, $\mathcal{S}^*$, or $\mathcal{K}$, then its n-th section is respectively convex, starlike, or close-to-convex in the disk $|z| < 1 - 3n^{-1}\log n$, for $n \geq 5$. Concerning the class $\mathcal{S}$, the first two authors in [?] proved the following

Theorem B. *Let $f \in \mathcal{S}$. Then every section $s_n(z)$ of f is starlike in the disk $|z| \leq 1/2$ for all $n \geq 47$.*

In this paper, we shall consider the class $\mathcal{G}(\alpha)$ and discuss various properties of it. A locally univalent function $f \in \mathcal{A}$ is said to belong to $\mathcal{G}(\alpha)$, for some $\alpha > 0$, if it satisfies the condition

$$\text{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) < 1+\frac{\alpha}{2}, \quad z\in\mathbb{D}. \tag{2}$$

In [?], Ozaki introduced the class $\mathcal{G}(1)\equiv\mathcal{G}$ and proved that functions in $\mathcal{G}$ are univalent in $\mathbb{D}$. Later in [?] Umezawa discussed a general version of this class. Moreover, functions in $\mathcal{G}$ are proved to be starlike in $\mathbb{D}$, see for eg. [?, Example 1, Equation (16)] and [?, Theorem 1] (see also [?]). Thus, the class $\mathcal{G}(\alpha)$ is included in $\mathcal{S}^*$ whenever $\alpha\in(0,1]$. It can be easily seen that functions in $\mathcal{G}(\alpha)$ are not necessarily univalent in $\mathbb{D}$ if $\alpha>1$. However, to our knowledge there is no sharp necessary coefficient bound on $|a_n|$ for functions in $\mathcal{G}(\alpha)$. In Theorem **??**, we solve this problem using the method of Rogosinski [?].

For $f\in\mathcal{A}$ of the form (**??**), the classical Fekete-Szegö coefficient functional

$$\Lambda_\lambda(f)=a_3-\lambda a_2^2$$

plays an important role in function theory. The problem of maximizing the absolute value of the functional $\Lambda_\lambda(f)$ is called the Fekete-Szegö problem, see [?]. For instance, if $f\in\mathcal{S}$ and $\lambda\in(0,1)$, then the Fekete-Szegö theorem [?] (see also [?, Theorem 3.8, p. 104]) gives that

$$|a_3-\lambda a_2^2|\le 1+2\exp(-2\lambda/(1-\lambda)) \quad \text{for } \lambda\in(0,1).$$

Equality holds for a function $f\in\mathcal{S}$ and with real coefficients. Allowing $\lambda\to 1-$, we have the elementary inequality $|a_3-a_2^2|\le 1$ for $f\in\mathcal{S}$. Solution to the Fekete-Szegö problem corresponding to various subclasses of $\mathcal{S}$ have been dealt by a number of authors. See for example [?, ?, ?, ?, ?] and the recent paper [?] on this problem for the class of concave functions.

The paper is organized as follows. First we prove sharp coefficient bounds for the moduli of the Taylor coefficients a_n of $f\in\mathcal{G}(\alpha)$. Secondly, we solve the Fekete-Szegö problem for functions in the class $\mathcal{G}(\alpha)$ with complex parameter λ. Thirdly, we present a convolution characterization for functions f to be in the class $\mathcal{G}(\alpha)$. As a consequence we obtain several sufficient coefficient conditions for f to be in $\mathcal{G}(\alpha)$. In Section **??**, we present a number of useful lemmas. Finally, in Section **??**, we use these lemmas to discuss the close-to-convexity and starlikeness of sections of $f\in\mathcal{G}(\alpha)$.

2. Coefficient Estimates and the Fekete-Szegö Functional

Theorem 1. *Let $f\in\mathcal{G}(\alpha)$ for some $0<\alpha\le 1$ and $f(z)=\sum_{n=1}^\infty a_nz^n$. Then*

$$|a_n|\le\frac{\alpha}{n(n-1)} \quad \text{for } n\ge 2.$$

Equality is attained for the function f_n such that $f_n'(z)=(1-z^{n-1})^{\alpha/(n-1)}$, $n\ge 2$.

Proof. Let $g(z)=zf'(z)$, where $f\in\mathcal{G}(\alpha)$. Then, according to (**??**), we need to consider functions g that fulfill the condition

$$\text{Re}\left(\frac{zg'(z)}{g(z)}\right)<\gamma=1+\frac{\alpha}{2}, \quad z\in\mathbb{D}.$$

This is equivalent to

$$\mathrm{Re}\left(\frac{\gamma - \frac{zg'(z)}{g(z)}}{\gamma - 1}\right) > 0, \quad z \in \mathbb{D}. \tag{3}$$

Since the function whose real part we have considered here is normalized such that its value at the origin equals 1, there exists an analytic function $\omega : \mathbb{D} \to \overline{\mathbb{D}}$ such that

$$\frac{\gamma - \frac{zg'(z)}{g(z)}}{\gamma - 1} = \frac{1 + z\omega(z)}{1 - z\omega(z)}. \tag{4}$$

Some obvious computations deliver

$$zg'(z) - g(z) = z\omega(z)(zg'(z) + (1 - 2\gamma)g(z)). \tag{5}$$

Now, let

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

Then $b_n = na_n$ for $n \geq 1$, where $a_1 = 1$. It suffices to find bounds for the moduli of the Taylor coefficients b_n, $n \geq 2$. In order to achieve this goal, we use Theorem 2.2 in [?] (compare also [?, ?, ?]). Using these methods, we immediately see that (??) implies for each $N \geq 2$ the inequality

$$\sum_{k=2}^{N} |(k-1)b_k|^2 \leq \sum_{k=1}^{N-1} |(k-1-\alpha)b_k|^2, \quad \alpha = 2(\gamma - 1).$$

Since $b_1 = 1$, we derive by mathematical induction the inequality

$$|b_n| \leq \frac{\alpha}{n-1} \tag{6}$$

for each $n \geq 2$. Equality in (??) can be achieved if and only if

$$b_m = 0, \quad m = 1, \ldots, n-1.$$

If we insert this into (??) and assume equality in (??), we recognize that this implies

$$\omega(z) = z^{n-2}, \quad n \geq 2.$$

Thus, the upper bound is attained for the function

$$g_n(z) = zf_n'(z) = z(1 - z^{n-1})^{\alpha/(n-1)}, \ n \geq 2.$$

The proof is completed. □

For a locally univalent analytic function f in $\mathbb{D}$, the pre-Schwarzian derivative T_f of f is defined by $T_f = f''/f'$, and we define the norm of T_f by

$$\|T_f\| = \sup_{z \in \mathbb{D}} (1 - |z|^2)|T_f(z)|.$$

This is indeed a norm with respect to the Hornich operations (see [?]). It is well known that if $\|T_f\| \leq 1$ then f is univalent in $\mathbb{D}$. On the converse part, every univalent function $f \in \mathcal{S}$ necessarily satisfies the condition $\|T_f\| \leq 6$ and the bound 6 is sharp (see [?]).

As a quick corollary to Theorem **??**, we can state the exact set of variability for the functional $(1-|z|^2)T_f(z)$, $f \in \mathcal{G}(\alpha)$, which essentially gives the sharp upper bound for the pre-Schwarzian norm $\|T_f\|$.

Corollary 1. *Let $\alpha \in (0,1]$ be fixed. Then the set of variability of the functional $(1-|z|^2)T_f(z)$, $f \in \mathcal{G}(\alpha)$, is the closed disk centered at the origin with radius 2α.*

Proof. Let $f \in \mathcal{G}(\alpha)$. According to the characterization (**??**) or (**??**), for functions $f \in \mathcal{G}(\alpha)$ (with $g = zf'$), we easily have

$$\frac{f''(z)}{f'(z)} = -\frac{\alpha\omega(z)}{1 - z\omega(z)}.$$

By the condition $|\omega(z)| \le 1$, this gives the estimate

$$\left|\frac{f''(z)}{f'(z)}\right| \le \frac{\alpha}{1-|z|}$$

and therefore,

$$(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \le \alpha(1+|z|) < 2\alpha.$$

The bound 2α is sharp as the function $f(z)$ for which $f'(z) = (1-z)^\alpha$ shows. $\square$

In particular, Corollary **??** shows that every $f \in \mathcal{G}$ necessarily satisfies the sharp inequality $\|T_f\| \le 2$.

Theorem 2. *Let $f \in \mathcal{G}(\alpha)$ for some $0 < \alpha \le 1$ and $f(z) = z + \sum_{n=2}^\infty a_n z^n$. Then, we have*

$$\left|a_3 - \lambda {a_2}^2\right| \le \begin{cases} \dfrac{\alpha}{6}\left|1-\alpha-\dfrac{3}{2}\alpha\lambda\right| & \text{for } \left|\lambda - \dfrac{2}{3\alpha}(1-\alpha)\right| \ge \dfrac{2}{3\alpha} \\ \dfrac{\alpha}{6} & \text{for } \left|\lambda - \dfrac{2}{3\alpha}(1-\alpha)\right| < \dfrac{2}{3\alpha}. \end{cases}$$

Equality in the Fekete-Szegö functional is attained in each case.

Proof. Let $f \in \mathcal{G}(\alpha)$, $f(z) = \sum_{n=1}^\infty a_n z^n$ with $a_1 = 1$, $zf'(z) = g(z)$ and $\gamma = 1 + (\alpha/2)$. Then (**??**) holds and is equivalent to

$$z^2 f''(z) = z\omega(z)(z^2 f''(z) - \alpha z f'(z)).$$

If in this formula, we let

$$\omega(z) = \sum_{k=0}^\infty c_k z^k,$$

then we get by comparing the coefficients of z^2 and z^3

$$2a_2 = -c_0\alpha \quad \text{and} \quad 3a_3 = -\frac{c_1}{2}\alpha + c_0(1-\alpha)a_2.$$

Using the first equation, the second becomes

$$3a_3 = -\frac{c_1}{2}\alpha - \frac{c_0^2}{2}\alpha(1-\alpha).$$

Using these two expressions, the Fekete-Szegö functional for the family $\mathcal{G}(\alpha)$ takes the form

$$|a_3 - \lambda a_2^2| = \frac{1}{6}\alpha |c_1 + \mu c_0^2|$$

where

$$\mu = (1-\alpha) - \frac{3}{2}\alpha\lambda.$$

It is well-known that $|c_0| \leq 1$ and $|c_1| \leq 1 - |c_0|^2$. Due to the second inequality, which is sharp, we get

$$|a_3 - \lambda a_2^2| \leq \frac{\alpha}{6}(1 - |c_0|^2 + |\mu|\,|c_0|^2).$$

Hence, the following sharp inequality is valid:

$$|a_3 - \lambda a_2^2| \leq \frac{\alpha}{6}|\mu| \quad \text{if } |\mu| \geq 1,$$

and the equality in this inequality is attained for $f'(z) = (1 - ze^{i\theta})^{\alpha}$, $\theta \in [0, 2\pi]$. In the other case, we have

$$|a_3 - \lambda a_2^2| \leq \frac{\alpha}{6} \quad \text{if } |\mu| < 1.$$

Equality in the last inequality is attained for $f'(z) = (1 - z^2 e^{i\theta})^{\alpha/2}$, $\theta \in [0, 2\pi]$. Finally, it is a simple exercise to see that $|\mu| < 1$ if and only

$$\left|\lambda - \frac{2}{3\alpha}(1-\alpha)\right| < \frac{2}{3\alpha}.$$

The proof is completed. □

If we let $\alpha = 1$, then the above theorem gives the following.

Corollary 2. *If $f \in \mathcal{G}$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then*

$$\left|a_3 - \lambda {a_2}^2\right| \leq \begin{cases} \dfrac{\lambda}{4} & \text{for } |\lambda| \geq \dfrac{2}{3} \\[2mm] \dfrac{1}{6} & \text{for } |\lambda| < \dfrac{2}{3}. \end{cases}$$

Equality in the Fekete-Szegö functional is attained in each case. In particular, one has the sharp inequality $|a_3 - {a_2}^2| \leq 1/4$ for $f \in \mathcal{G}$.

3. Convolution Characterization

If f, g are analytic in $\mathbb{D}$ with

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

then the Hadamard product (or convolution) of f and g is defined by the function

$$(f \star g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Clearly, $f \star g$ is analytic in $\mathbb{D}$. In view of the Hadamard convolution, it is now possible to present a convolution characterization for functions in the class $\mathcal{G}(\alpha)$.

Theorem 3. *Let $0 < \alpha \le 1$. Then, $f \in \mathcal{G}(\alpha)$ if and only if*

$$\frac{f(z)}{z} \star \frac{1 - ((2/\alpha)(x+1)+1)z}{(1-z)^3} \neq 0 \tag{7}$$

for all $|z| < 1$ and for all x with $|x| = 1$. Equivalently, this holds if and only if

$$\sum_{n=1}^{\infty} A_n z^{n-1} \neq 0 \quad (z \in \mathbb{D},\ |x| = 1) \tag{8}$$

where $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and

$$A_n = n a_n \big(1 - ((x+1)/\alpha)(n-1)\big) \quad (n \ge 1,\ a_1 = 1).$$

Proof. We recall that $f \in \mathcal{G}(\alpha)$ if and only if $\operatorname{Re} P(z) > 0$ in $\mathbb{D}$, where

$$P(z) = 1 - \frac{2}{\alpha} \frac{z f''(z)}{f'(z)}.$$

We note that P is analytic in $\mathbb{D}$ with $P(0) = 1$. Thus, $f \in \mathcal{G}(\alpha)$ is equivalent to

$$P(z) \neq \frac{x-1}{x+1} \quad (z \in \mathbb{D},\ |x| = 1,\ x \neq -1)$$

which, by a simplification, is same as writing

$$(x+1)g'(z) - (\alpha + x + 1)\frac{g(z)}{z} \neq 0 \tag{9}$$

where $g(z) = zf'(z)$. Recall that

$$\frac{g(z)}{z} = \frac{g(z)}{z} \star \frac{1}{1-z} \quad \text{and} \quad zg'(z) = g(z) \star \frac{z}{(1-z)^2}.$$

Using these identities, $g(z) = zf'(z)$, (**??**) gives that $f \in \mathcal{G}(\alpha)$ if and only if

$$f'(z) \star \left((x+1)\frac{1}{(1-z)^2} - (\alpha + x + 1)\frac{1}{1-z} \right) \neq 0.$$

Thus, (**??**) is equivalent to

$$\frac{1}{z}\left[zf'(z) \star \left((x+1)\frac{z}{(1-z)^2} - (\alpha + x + 1)\frac{z}{1-z} \right) \right] \neq 0.$$

Since $zf'(z) \star \phi(z) = f(z) \star z\phi'(z)$, we may write the last relation as

$$\frac{1}{z}\left[f(z) \star \left((x+1)\frac{z(1+z)}{(1-z)^3} - (\alpha + x + 1)\frac{z}{(1-z)^2} \right) \right] \neq 0.$$

After some simplification the above takes the equivalent form given by (**??**), i.e.

$$\frac{f(z)}{z} \star \frac{q(z)}{z} \neq 0, \quad q(z) = \frac{\alpha z - (2(x+1)+\alpha)z^2}{(1-z)^3}.$$

To obtain the series formulation of it, it suffices to observe that

$$q(z) = \sum_{n=1}^{\infty} n\big((x+1)n - (\alpha + x + 1)\big) z^n.$$

The proof is completed. □

Theorem 4. *Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfies any one of the following conditions for some $\alpha \in (0,1]$:*

(a) $\sum_{n=2}^{\infty} n(2(n-1)-\alpha)|a_n| \le \alpha$,

(b) $\sum_{n=2}^{\infty} |n(n-1-\alpha)a_n - (n-1)(n-2-\alpha)a_{n-1}| + (n-1)|na_n - (n-2)a_{n-1}| \le \alpha$,

(c) $\sum_{n=2}^{\infty} |n(n-1-\alpha)a_n - (n-2)(n-3-\alpha)a_{n-2}| + |(n-1)na_n - (n-3)(n-2)a_{n-2}| \le \alpha$.

Then $f \in \mathcal{G}(\alpha)$.

Proof. According to the convolution criterion of functions in $\mathcal{G}(\alpha)$ and the notation of Theorem **??**, it suffices to show that

$$1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0 \tag{10}$$

in $\mathbb{D}$ for $|x| = 1$. Here

$$A_n = -\frac{na_n}{\alpha}\big(n - (1+\alpha) + (n-1)x\big) \quad (n \ge 2).$$

Let us first assume that (a) holds. We observe that the condition (**??**) obviously holds, if $\sum_{n=2}^{\infty} |A_n| \le 1$. Indeed, for $|x| = 1$, the triangle inequality gives

$$\sum_{n=2}^{\infty} |A_n| \le \sum_{n=2}^{\infty} \frac{n|a_n|}{\alpha}\big(n - (1+\alpha) + (n-1)\big) = \frac{1}{\alpha}\sum_{n=2}^{\infty} n|a_n|\big(2n - 2 - \alpha\big)$$

and the condition (a) implies that $\sum_{n=2}^{\infty} |A_n| \le 1$.

Next we assume (b). If we multiply (**??**) by a nonvanishing factor $1 - z$, $z \in \mathbb{D}$, it can be easily seen that $\sum_{n=2}^{\infty} |A_n - A_{n-1}| \le 1$ infers that $f \in \mathcal{G}(\alpha)$. Clearly, for all $|x| = 1$, we have

$$\alpha|A_n - A_{n-1}| \le |na_n(n-(1+\alpha)) - (n-1)a_{n-1}(n-1-(1+\alpha))| + (n-1)|na_n - (n-2)a_{n-1}|$$

and thus, the assertion follows under the condition (b).

The final case follows if we multiply the inequality (**??**) by $1 - z^2$ and proceed with the above method of proof. Indeed

$$(1 - z^2)\left(1 + \sum_{n=2}^{\infty} A_n z^{n-1}\right) = 1 + \sum_{n=2}^{\infty} (A_n - A_{n-2}) z^{n-1}$$

and

$$\alpha|A_n - A_{n-2}| \le |n(n-1-\alpha)a_n - (n-2)(n-3-\alpha)a_{n-2}| + |(n-1)na_n - (n-3)(n-2)a_{n-2}|.$$

Clearly, a computation shows that the condition (c) implies that

$$\sum_{n=2}^{\infty} |A_n - A_{n-2}| \leq 1$$

and hence, $f \in \mathcal{G}(\alpha)$. ☐

We remark that one can obtain several new sufficient conditions if we multiply the inequality (**??**) by a suitable non-vanishing polynomials in $\mathbb{D}$, such as $(1-z)^2$ and $1+z+z^2$, and proceed with the above method of proof. The case $\alpha = 1$ of Theorem **??** gives the following simple form.

Corollary 3. *Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfies any one of the following conditions:*

(a) $\displaystyle\sum_{n=2}^{\infty} n(2n-3)|a_n| \leq 1,$

(b) $\displaystyle\sum_{n=2}^{\infty} |n(n-2)a_n - (n-1)(n-3)a_{n-1}| + (n-1)|na_n - (n-2)a_{n-1}| \leq 1,$

(c) $\displaystyle\sum_{n=2}^{\infty} |n(n-2)a_n - (n-2)(n-4)a_{n-2}| + |(n-1)na_n - (n-3)(n-2)a_{n-2}| \leq 1.$

Then $f \in \mathcal{G}$.

4. Some lemmas

Lemma 1. *Let $f \in \mathcal{G}(\alpha)$ and*

$$\frac{1}{f'(z)} = 1 + d_1 z + d_2 z^2 + \cdots \tag{11}$$

for some complex coefficients d_n. Then the inequality

$$|d_n| \leq (-1)^n \binom{-\alpha}{n} = \frac{1}{n!} \prod_{k=1}^{n} (\alpha + k - 1) \tag{12}$$

holds for each $n \in \mathbb{N}$. Equality is attained for $f'(z) = (1-z)^{\alpha}$.

Proof. As before, we let $g(z) = zf'(z)$ and consider the function u defined by

$$u(z) = \frac{z^2}{g(z)} = \frac{z}{f'(z)} = z + \sum_{k=1}^{\infty} d_k z^{k+1}.$$

Since

$$\mathrm{Re}\left(\frac{zu'(z)}{u(z)}\right) = \mathrm{Re}\left(2 - \frac{zg'(z)}{g(z)}\right) > 1 - \frac{\alpha}{2}$$

for $f \in \mathcal{G}(\alpha)$, we get as in the proof of Theorem **??** that there exists an analytic function $\omega : \mathbb{D} \to \overline{\mathbb{D}}$ such that

$$\frac{1}{\alpha}\left(2\frac{zu'(z)}{u(z)} - 2 + \alpha\right) = \frac{1 + z\omega(z)}{1 - z\omega(z)}.$$

After some obvious computations this implies

$$zu'(z) - u(z) = z\omega(z)(zu'(z) - (1-\alpha)u(z)).$$

Use of the same method as in the proof of Theorem **??** leads us to the inequalities

$$\sum_{k=1}^{n} k^2|d_k|^2 \leq \alpha^2 + \sum_{k=1}^{n-1}(k+\alpha)^2|d_k|^2$$

for each $n \in \mathbb{N}$. The last inequality is equivalent to the inequality

$$n^2|d_n|^2 \leq \alpha^2 + \alpha\sum_{k=1}^{n-1}(2k+\alpha)|d_k|^2$$

and the assertion (**??**) follows by mathematical induction. □

To prove the next lemma, we need the following result whose proof is a simple exercise, and so omit its details.

Lemma 2. *Let $k \in \mathbb{N}$ and $1 \leq i \leq k, n_j \in \mathbb{N}, j = 1, \ldots, i$, be such that $\sum_{j=1}^{i} n_j = k$. Then*

$$\prod_{j=1}^{i}(n_j+1) \geq k+1.$$

Lemma 3. *For $\alpha \in (0,1)$, let $h'(z) = (1-z)^\alpha$. If*

$$\frac{z}{h(z)} = 1 + \sum_{n=1}^{\infty} \delta_n z^n,$$

then

$$0 < \delta_n \leq \frac{1}{(n+1)!}\prod_{k=1}^{n}(\alpha+k-1) \leq \frac{\alpha}{n+1} \tag{13}$$

for each $n \in \mathbb{N}$.

Proof. The third inequality in (**??**) is obvious and so, we now prove the first and the second one. For $\alpha \in (0,1)$, we consider

$$h'(z) = (1-z)^\alpha = 1 - \sum_{n=1}^{\infty}\gamma_n z^n,$$

where

$$\gamma_n = (-1)^{n-1}\binom{\alpha}{n} > 0.$$

By integration this formula gives

$$h(z) = z - \sum_{n=1}^{\infty}\frac{\gamma_n}{n+1}z^{n+1}$$

so that

$$\frac{z}{h(z)} = \frac{1}{1-\sum_{n=1}^{\infty}\frac{\gamma_n}{n+1}z^n} = 1+\sum_{n=1}^{\infty}\delta_n z^n.$$

To prove (**??**), we use the abbreviations

$$h'(z) = 1-F(z) \quad \text{and} \quad \frac{h(z)}{z} = 1-G(z),$$

which imply

$$\frac{1}{h'(z)} = 1+\sum_{k=1}^{\infty}F(z)^k = 1+\sum_{n=1} d_n z^n \quad \text{and} \quad \frac{z}{h(z)} \;=\; 1+\sum_{k=1}^{\infty}G(z)^k.$$

From these representations we see that the d_n can be computed as polynomials with positive coefficients in the γ_n, where the monomials are of the form

$$\prod_{j=1}^{i}\gamma_{n_j}.$$

Therein i and the $n_j, j = 1, \ldots, i$ are to be chosen as in Lemma **??**. If we compute the δ_n in an analogous way using the function G, we get analogous polynomials, but the monomials are of the form

$$\prod_{j=1}^{i}\frac{\gamma_{n_j}}{n_j+1}.$$

This immediately proves the first inequality in (**??**). Since Lemma **??** implies

$$\prod_{j=1}^{i}\frac{\gamma_{n_j}}{n_j+1} \le \frac{1}{k+1}\prod_{j=1}^{i}\gamma_{n_j},$$

we have proven the second one as well.

Comparison of this expansions with the expansion of $1/h'$ delivers (**??**) for each $n \in \mathbb{N}$. ☐

Lemma 4. *Suppose that $f \in \mathcal{G}(\alpha)$ for some $\alpha \in (0,1]$, and $s_n(z)$ is its n-th partial sum. Assume that $|1/f'(z)| \le M$ in $\mathbb{D}$ for some $M > 1$. Then for each $n \ge 2$*

$$\left|\frac{s_n'(z)}{f'(z)} - 1\right| \le |z|^n\left(\frac{\alpha}{n} + A_n\frac{|z|}{1-|z|}\right), \ |z| = r < 1,$$

where $A_n = \left(\frac{\alpha\pi}{\sqrt{6}}+1\right)\sqrt{M^2-1}$.

Proof. We follow the method of proof of [**?**] with necessary modifications. For $f \in \mathcal{G}(\alpha)$, we let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ so that

$$s_n(z) = z + a_2z^2 + a_3z^3 + \cdots + a_nz^n.$$

Since $f \in \mathcal{G}(\alpha)$ is univalent (and starlike) in $\mathbb{D}$, $f'(z)$ is non-vanishing in $\mathbb{D}$ and hence, $1/f'(z)$ can be represented in the form (**??**) so that

$$(1 + 2a_2z + 3a_3z^2 + \cdots)(1 + d_1z + d_2z^2 + \cdots) \equiv 1.$$

Note that $2a_2 = -d_1$ and from the last relation we see that

$$\sum_{k=1}^{m-1}(m-k)a_{m-k}d_k + ma_m = 0 \quad (m = 2, 3, \ldots;\ a_1 = 1).$$

By using the representation for the partial sum $s_n(z)$, we obtain that

$$\frac{s_n'(z)}{f'(z)} \equiv 1 + c_n z^n + c_{n+1}z^{n+1} + \cdots ,$$

where

$$c_n = na_n d_1 + (n-1)a_{n-1}d_2 + \cdots + a_1 d_n.$$

The previous relation for $m = n+1$ shows that $c_n = -(n+1)a_{n+1}$ and more generally,

$$c_m = na_n d_{m-n+1} + (n-1)a_{n-1}d_{m-n+2} + \cdots + 2a_2 d_{m-1} + d_m$$

for $m = n+1, n+2, \ldots$. By Theorem **??**, one has the inequality $|a_n| \leq \alpha/(n(n-1))$ for all $n \geq 2$, and therefore, we have that for $m \geq n+1$

$$|c_m - d_m| \leq \alpha \sum_{k=1}^{n-1} \frac{|d_{m-n+k}|}{n-k} = \alpha \sum_{k=1}^{n-1} \frac{|d_{m-k}|}{k}. \tag{14}$$

By assumption $|1/f'(z)| \leq M$ for $z \in \mathbb{D}$. Hence for $0 < r < 1$ we have from the series representation of $1/f'(z)$ that

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1}{f'(re^{i\theta})}\right|^2 d\theta = 1 + \sum_{k=1}^{\infty} |d_k|^2 r^{2k} \leq M^2$$

which, by allowing $r \to 1^-$, shows that

$$\sum_{k=1}^{\infty} |d_k|^2 \leq M^2 - 1. \tag{15}$$

In view of the Cauchy-Schwarz inequality and the last inequality, (**??**) reduces to

$$|c_m - d_m| \leq \alpha \left(\sum_{k=1}^{n-1} \frac{1}{k^2}\right)^{1/2} \left(\sum_{k=1}^{n-1} |d_{m-k}|^2\right)^{1/2} \leq \alpha\sqrt{(\pi^2/6)(M^2-1)}$$

for $m \geq n+1$. Moreover, by (**??**) we have $|d_k| \leq M^2 - 1$ for each $k \geq 1$. Using this, the last inequality gives

$$|c_m| \leq \left(\frac{\alpha\pi}{\sqrt{6}} + 1\right)\sqrt{M^2-1} = A_n$$

for $m\geq n+1$. This inequality together with the fact that $|c_n|=|(n+1)a_{n+1}|\leq \alpha/n$ (by Theorem **??**), gives that for $|z|=r<1$,

$$\begin{aligned}\left|\frac{s_n'(z)}{f'(z)}-1\right| &= \left|c_nz^n+c_{n+1}z^{n+1}+\cdots\right|\\ &\leq |c_n|\,|z|^n+|c_{n+1}|\,|z|^{n+1}+\cdots\\ &\leq |z|^n\left(\frac{\alpha}{n}+A_n\frac{|z|}{1-|z|}\right)\end{aligned}$$

for $n\geq 2$. This completes the proof of Lemma **??**. □

Lemma 5. *Let $f\in\mathcal{G}(\alpha)$ for some $\alpha\in(0,1]$ and $s_n(z)$ be its n-th partial sum. Then for each $r\in(0,1)$ and $n\geq 2$, we have*

$$\left|\frac{s_n'(z)}{f'(z)}-1\right|<|z|^n\left(\frac{\alpha}{n}+\left(\frac{\alpha\pi}{\sqrt{6}}+1\right)\frac{\sqrt{1-(1-r)^{2\alpha}}}{(1-r)^\alpha r^n}\frac{|z|}{r-|z|}\right)\quad \textit{for } |z|<r. \tag{16}$$

Proof. Suppose that $f\in\mathcal{G}(\alpha)$. Then

$$\frac{1}{f'(z)}\prec\frac{1}{(1-z)^\alpha},\quad z\in\mathbb{D},$$

where $\prec$ denotes the usual subordination ([**?**, **?**]). Therefore,

$$\left|\frac{1}{f'(z)}\right|\leq\frac{1}{(1-r)^\alpha}=:M(r)\ \text{ for } |z|=r<1. \tag{17}$$

Following the proof and the notation of Lemma **??**, one has

$$\sum_{k=1}^{\infty}|d_k|^2r^{2k}\leq M(r)^2-1 \tag{18}$$

and therefore, (**??**) may be rewritten as

$$|c_m-d_m|\leq\alpha\sum_{k=1}^{n-1}\frac{|d_{m-k}|}{k}=\alpha\sum_{k=1}^{n-1}\left(\frac{1}{kr^{m-k}}\right)\left(|d_{m-k}|r^{m-k}\right)$$

for any arbitrary fixed $r\in(0,1)$. Thus, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|c_m-d_m|^2 &\leq \alpha^2\left(\sum_{k=1}^{n-1}\frac{1}{k^2r^{2(m-k)}}\right)\left(\sum_{k=1}^{n-1}|d_{m-k}|^2r^{2(m-k)}\right)\\ &\leq \alpha^2\left(\frac{1}{r^{2m}}\sum_{k=1}^{n-1}\frac{1}{k^2}\right)\left(M(r)^2-1\right)\\ &\leq \frac{\alpha^2}{r^{2m}}\frac{\pi^2}{6}\left(M(r)^2-1\right)\end{aligned}$$

which is true for each fixed $r\in(0,1)$, and for $m\geq n+1$. Further, because (see (**??**))

$$|d_k|r^k\leq\sqrt{M(r)^2-1}\ \text{ for each } k\geq 1,$$

it follows from the last inequality that for $m \geq n+1$

$$|c_m| \leq |d_m| + \frac{\alpha}{r^m}\frac{\pi}{\sqrt{6}}\sqrt{M(r)^2-1} \leq \frac{1}{r^m}\left(\frac{\alpha\pi}{\sqrt{6}}+1\right)\left(\sqrt{M(r)^2-1}\right).$$

As in the proof of Lemma **??**, using the above estimate, we easily have

$$\left|\frac{s_n'(z)}{f'(z)}-1\right| < |z|^n\left(\frac{\alpha}{n}+\frac{1}{r^n}\left(\frac{\alpha\pi}{\sqrt{6}}+1\right)\left(\sqrt{M(r)^2-1}\right)\frac{|z|/r}{1-(|z|/r)}\right)$$

for $|z|<r$. Finally, the proof of the theorem follows if we substitute in the above the expression $M(r)=(1-r)^{-\alpha}$ given by (**??**). □

Given complex numbers a, b, and c with $c \neq 0, 1, 2, \ldots$, let

$$F(a,b;c;z) := \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \quad z \in \mathbb{D},$$

denote the Gaussian hypergeometric function, where $(a)_k$ denotes the Pochhammer symbol $(a)_0 = 1$, $(a)_k := a(a+1)\cdots(a+k-1)$ for $k \in \mathbb{N}$. When a is neither zero nor a negative integer, then using the definition of the Gamma function we write

$$(a)_k = \frac{\Gamma(k+a)}{\Gamma(a)}.$$

The following well-known formula

$$F(a,b;c;1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}\,(c-a-b) > 0,$$

will be used and the function $F(a,b;c;z)$ is bounded if $\mathrm{Re}\,(c-a-b)>0$.

Lemma 6. *Suppose that $f \in \mathcal{G}(\alpha)$ for some $\alpha \in (0,1]$, and $s_n(z)$ is its n-th partial sum. Then for each $n \geq 2$ we have the following:*

(1) *for $\alpha = 1$,*

$$\left|\frac{s_n(z)}{f(z)}-1\right| < |z|^n\left(\frac{1}{n(n+1)}+R\frac{|z|}{1-|z|}\right), \ |z|=r<1,$$

where $R = \dfrac{1}{\sqrt{3}}+\dfrac{\pi^2}{3\sqrt{30}}$.

(2) *for $\alpha \in (0,1)$,*

$$\left|\frac{s_n(z)}{f(z)}-1\right| < |z|^n\left(\frac{\alpha}{n(n+1)}+R(\alpha)\frac{|z|}{1-|z|}\right), \ |z|=r<1,$$

where

$$R(\alpha) = \left(1+\frac{\alpha\pi^2}{3\sqrt{10}}\right)\frac{\sqrt{F(\alpha-1,\alpha-1;1;1)-(\alpha-1)^2-1}}{1-\alpha}.$$

Proof. As in the proof of Lemma **??**, we let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ so that $s_n(z) = z+a_2z^2+a_3z^3+\cdots+a_nz^n$. Since functions in $\mathcal{G}(\alpha)$, $\alpha \in (0, 1]$, are univalent, each $f \in \mathcal{G}(\alpha)$ can be written in the form

$$\frac{z}{f(z)} = 1 + b_1z + b_2z^2 + \cdots \tag{19}$$

for some complex coefficients b_n $(n \geq 1)$. In view of this observation and the other form of representation for f, it follows that

$$(1 + a_2z + a_3z^2 + \cdots)(1 + b_1z + b_2z^2 + \cdots) \equiv 1.$$

Comparing the powers of z on both sides, we have

$$\sum_{k=1}^{m-1} b_k a_{m-k} + a_m = 0 \quad (m = 2, 3, \ldots;\ a_1 = 1). \tag{20}$$

Using the representation for the partial sum $s_n(z)$ of f and (**??**), we obtain that

$$\begin{aligned} \frac{s_n(z)}{f(z)} &= (1 + a_2z + a_3z^2 + \cdots + a_nz^{n-1})(1 + b_1z + b_2z^2 + \cdots) \\ &\equiv 1 + c_nz^n + c_{n+1}z^{n+1} + \cdots, \end{aligned}$$

where

$$c_n = b_1a_n + b_2a_{n-1} + \cdots + b_na_1. \tag{21}$$

By (**??**), we observe that the coefficients of z^k in the above expansion for $k = 1, 2, \ldots, n-1$ vanish. Equation (**??**) for $m = n+1$ shows that $c_n = -a_{n+1}$. Also

$$c_m = b_{m-n+1}a_n + b_{m-n+2}a_{n-1} + \cdots + b_ma_1 \tag{22}$$

for $m = n+1, n+2, \ldots$. By Theorem **??**, $|a_n| \leq \alpha/(n(n-1))$ for all $n \geq 2$, and therefore, for $m \geq n+1$, we have

$$|c_m - b_m| \;\leq\; \frac{\alpha}{n(n-1)}|b_{m-n+1}| + \frac{\alpha}{(n-1)(n-2)}|b_{m-n+2}| + \cdots + \frac{\alpha}{2}|b_{m-1}|.$$

Using the classical Cauchy-Schwarz inequality, it follows that for $m \geq n+1$,

$$|c_m - b_m|^2 \leq \alpha^2 \left(\sum_{k=1}^{n-1} \frac{1}{(n+1-k)^2(n-k)^2}\right)\left(\sum_{k=1}^{n-1} |b_{m-n+k}|^2\right) = \alpha^2 AB$$

where

$$A = \sum_{k=1}^{n-1} \frac{1}{(k+1)^2k^2} \quad \text{and} \quad B = \sum_{k=1}^{n-1} |b_{m-n+k}|^2.$$

For $f \in \mathcal{G}(\alpha)$ we have $f'(z) \prec (1-z)^\alpha$ which in turn gives by the subordination result that

$$\frac{f(z)}{z} \prec \frac{1-(1-z)^{\alpha+1}}{z(\alpha+1)}, \quad z \in \mathbb{D}.$$

When f is of the form (**??**), it is convenient to write the last subordination relation in the form

$$\frac{z}{f(z)} \prec \frac{z(\alpha+1)}{1-(1-z)^{\alpha+1}} = 1+\sum_{k=1}^{\infty} d_k z^k. \tag{23}$$

By using the Rogosinski theorem (see [**?**, Theorem 6.2, p. 192]) we obtain that

$$\sum_{k=1}^{n} |b_k|^2 \leq \sum_{k=1}^{n} |d_k|^2. \tag{24}$$

Case 1. Let $\alpha = 1$. In this case (**??**) takes the form

$$\frac{z}{f(z)} \prec \frac{1}{1-(1/2)z} = 1+\sum_{k=1}^{\infty} \frac{1}{2^k} z^k$$

so that $d_k = 2^{-k}$ for $k \geq 1$. By this observation, (**??**) reduces to

$$\sum_{k=1}^{n} |b_k|^2 \leq \sum_{k=1}^{n} |d_k|^2 = \sum_{k=1}^{n} \frac{1}{4^k} = \frac{1}{3}\left(1-\frac{1}{4^n}\right)$$

which implies that

$$B \leq \sum_{k=1}^{\infty} |b_k|^2 \leq \frac{1}{3}.$$

Thus, $B \leq 1/3$. Also, $|b_m| \leq 1/\sqrt{3}$ for each $m \geq 1$. On the other hand for the sum A, we observe that

$$A = \sum_{k=1}^{n-1} \frac{1}{(k+1)^2k^2} \leq \sum_{k=1}^{n-1} \frac{1}{k^4} < \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

Thus for $m \geq n+1$ we have

$$|c_m - b_m| \leq \sqrt{AB} < \frac{\pi^2}{\sqrt{270}} \quad \text{for } m \geq n+1$$

so that

$$|c_m| \leq |b_m| + \frac{\pi^2}{\sqrt{270}} \leq \frac{1}{\sqrt{3}} + \frac{\pi^2}{3\sqrt{30}} = R \quad \text{for } m \geq n+1.$$

This inequality, together with the fact that $|c_n| = |a_{n+1}| \leq \frac{1}{(n+1)n}$ (for the case $\alpha = 1$), gives that for $|z| = r < 1$,

$$\begin{aligned} \left|\frac{s_n(z)}{f(z)} - 1\right| &\leq |c_n|\,|z|^n + |c_{n+1}|\,|z|^{n+1} + \cdots \\ &< \frac{1}{(n+1)n}|z|^n + R\left(|z|^{n+1} + |z|^{n+2} + \cdots\right) \\ &= |z|^n\left(\frac{1}{(n+1)n} + R\frac{|z|}{1-|z|}\right) \end{aligned}$$

for $n \geq 2$. The proof of Case 1 is done.

Case 2. Let $\alpha \in (0,1)$. Then, $f'(z) \prec h'(z) = (1-z)^\alpha$. According to Lemma **??**, (**??**) takes the form

$$\frac{z}{f(z)} \prec \frac{z}{h(z)} = 1 + \sum_{n=1}^{\infty} \delta_n z^n,$$

where δ_n satisfies the inequality (**??**) for $n \geq 1$, i.e.

$$\delta_n \leq \frac{(\alpha)_n}{(n+1)!}.$$

As in the proof of Case 1, it follows that

$$B \leq \sum_{k=1}^{\infty} |b_k|^2 \leq \sum_{k=1}^{\infty} |\delta_k|^2$$

where

$$\begin{aligned} \sum_{k=1}^{\infty} |\delta_k|^2 &\leq \sum_{k=1}^{\infty} \frac{(\alpha)_k(\alpha)_k}{(1)_{k+1}(1)_{k+1}} \\ &= \frac{1}{(\alpha-1)^2} \sum_{k=2}^{\infty} \frac{(\alpha-1)_k(\alpha-1)_k}{(1)_k(1)_k} \\ &= \frac{1}{(\alpha-1)^2}[F(\alpha-1,\alpha-1;1;1) - (\alpha-1)^2 - 1]. \end{aligned}$$

Thus,

$$B \leq \frac{1}{(\alpha-1)^2}[F(\alpha-1,\alpha-1;1;1) - (\alpha-1)^2 - 1]. \tag{25}$$

For $\alpha \in (0,1)$, we easily have

$$\frac{(\alpha)_k}{(k+1)!} \leq \frac{\alpha}{k+1}$$

and therefore, we can obtain the estimate

$$B \leq \alpha^2 \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} = \alpha^2 \left(\frac{\pi^2}{6} - 1\right)$$

although we do not use it here. However, using (**??**) and the estimate $A < \pi^4/(90)$, it follows that

$$|c_m - b_m| \leq \alpha\sqrt{AB} < \left(\frac{\sqrt{F(\alpha-1,\alpha-1;1;1) - (\alpha-1)^2 - 1}}{1-\alpha}\right) \frac{\alpha\pi^2}{\sqrt{90}}$$

for $m \geq n+1$. Moreover, (**??**) also gives

$$|b_m| \leq \frac{\sqrt{F(\alpha-1,\alpha-1;1;1) - (\alpha-1)^2 - 1}}{1-\alpha}$$

and thus, for $m \geq n+1$, we have

$$|c_m| \leq |b_m| + \alpha\sqrt{AB} < \left(1 + \frac{\alpha\pi^2}{3\sqrt{10}}\right) \frac{\sqrt{F(\alpha-1,\alpha-1;1;1) - (\alpha-1)^2 - 1}}{1-\alpha}.$$

The desired inequality in the statement follows as in the proof of Case 1. □

5. Injectivity and starlikeness of sections of $f \in \mathcal{G}(\alpha)$

Theorem 5. *Let $f \in \mathcal{G}(\alpha)$ for some $\alpha \in (0,1]$, and $s_n(z)$ be its n-th partial sum. Then, $\mathrm{Re}\,\{s_n'(z)\} > 0$ in the disk $|z| < 1/2$ for $n \geq N$, where N is the smallest natural number such that $\sin^{-1}(K_\alpha) \leq (3-\alpha)\pi/6$, where*

$$K_\alpha = \frac{1}{2^n}\left(\frac{\alpha}{n} + \left(\frac{\alpha\pi}{\sqrt{6}} + 1\right) 3\sqrt{9^\alpha - 1}\left(\frac{3}{2}\right)^n\right). \tag{26}$$

Here the natural number N depends on the values of α.

Proof. Let $f \in \mathcal{G}(\alpha)$. Then, if we put $r = 2/3$ in (**??**), then (**??**) gives

$$\left|\frac{s_n'(z)}{f'(z)} - 1\right| < |z|^n \left(\frac{\alpha}{n} + \left(\frac{\alpha\pi}{\sqrt{6}} + 1\right)\sqrt{9^\alpha - 1}\left(\frac{3}{2}\right)^n \frac{3|z|}{2-3|z|}\right) \quad \text{for } |z| < 2/3. \tag{27}$$

The inequality (**??**) for $|z| = 1/2$ together with the maximum modulus principle give that

$$\left|\frac{s_n'(z)}{f'(z)} - 1\right| < K_\alpha \quad \text{for } |z| < 1/2$$

and from this, we easily have

$$\max_{|z|=1/2}\left|\arg\frac{s_n'(z)}{f'(z)}\right| \leq \sin^{-1}(K_\alpha). \tag{28}$$

Moreover, as $f'(z) \prec (1-z)^\alpha$, it follows that

$$\max_{|z|=1/2} |\arg f'(z)| \leq \alpha \sin^{-1}\left(\frac{1}{2}\right) = \frac{\alpha\pi}{6}. \tag{29}$$

The last two inequalities together with the maximum modulus principle gives that

$$|\arg s_n'(z)| \leq |\arg f'(z)| + \left|\arg\frac{s_n'(z)}{f'(z)}\right| < \frac{\alpha\pi}{6} + \sin^{-1}(K_\alpha) \quad \text{for } |z| < \frac{1}{2}$$

and thus,

$$|\arg s_n'(z)| < \frac{\pi}{2}$$

holds if $\sin^{-1}(K_\alpha) \leq (3-\alpha)\pi/6$. The desired conclusion follows. □

The case $\alpha = 1$ simplifies to

Corollary 4. *Let $f \in \mathcal{G}$ and $s_n(z)$ be its n-th partial sum. Then, $\mathrm{Re}\,\{s_n'(z)\} > 0$ in the disk $|z| < 1/2$ for $n \geq 11$. In particular, $s_n'(z)$ is close-to-convex (and hence univalent) in $|z| < 1/2$ for $n \geq 11$.*

Proof. Set $\alpha = 1$ in Theorem **??**. Then, $\mathrm{Re}\, s_n'(z) > 0$ for $|z| < 1/2$ whenever $\sin^{-1}(K_1) \leq \pi/3$ holds, where K_1 is given by (**??**). However, the last inequality is easily seen to be true for all $n \geq 11$, and hence, $s_n(z)$ is close-to-convex for $|z| < 1/2$. □

Corollary **??** improves the result of Obradović and S. Ponnusamy [**?**] in which this result was proved for $n \geq 13$. Further, the values of $N = N(\alpha)$ in Theorem **??** for various choices of α in the interval $(0, 1)$ can be computed using Mathematica. For example, for

$$\alpha = 5/6, 4/5, 3/4, 2/3, 1/2, 1/3, 1/4, 1/5, 1/6,$$

the corresponding values of $N = N(\alpha)$ are given by

$$N(\alpha) = 10, 10, 9, 9, 7, 6, 5, 4, 4,$$

respectively.

In order to prove the next two results we need some preparations. For example, if $f \in \mathcal{G}(\alpha)$, then we have (see eg. [**?**, Theorem 1])

$$\frac{zf'(z)}{f(z)} \prec g(z) = \frac{(1+\alpha)(1-z)}{1+\alpha-z}, \quad z \in \mathbb{D}.$$

We see that the function g above is univalent in $\mathbb{D}$ and maps $\mathbb{D}$ onto the disk

$$\left| w - \frac{\alpha+1}{\alpha+2} \right| < \frac{\alpha+1}{\alpha+2}.$$

Further, it is a simple exercise to see that g maps the circle $|z| = r$ onto the circle

$$\left| w - \frac{(\alpha+1)(\alpha+1-r^2)}{(\alpha+1)^2 - r^2} \right| = \frac{(\alpha+1)\alpha r}{(\alpha+1)^2 - r^2}$$

and so, by a computation, we see that for $f \in \mathcal{G}(\alpha)$ we obtain that

$$\left| \arg \frac{zf'(z)}{f(z)} \right| \leq \sin^{-1}\left(\frac{\alpha r}{\alpha + 1 - r^2} \right), \quad |z| = r < 1.$$

In particular this gives

$$\left| \arg \frac{zf'(z)}{f(z)} \right| \leq \sin^{-1}\left(\frac{2\alpha}{4\alpha+3} \right) \quad \text{for } |z| \leq 1/2. \tag{30}$$

Theorem 6. *Let $f \in \mathcal{G}$. Then for $n \geq 11$, every section $s_n(z)$ of f is starlike in the disk $|z| < 1/2$.*

Proof. Let $f \in \mathcal{G}$. Inequality (**??**) in particular for $\alpha = 1$ gives

$$\left| \arg \frac{zf'(z)}{f(z)} \right| \leq \sin^{-1}\left(\frac{2}{7} \right) \quad \text{for } |z| \leq 1/2. \tag{31}$$

As in the proof of Theorem **??**, Case 1 of Lemma **??** gives that

$$\left| \frac{s_n(z)}{f(z)} - 1 \right| < \frac{1}{2^n}\left(\frac{1}{n(n+1)} + \frac{1}{\sqrt{3}} + \frac{\pi^2}{3\sqrt{30}} \right) =: N_2 \quad \text{for } |z| \leq 1/2$$

so that

$$\max_{|z|=1/2}\left|\arg\frac{s_n(z)}{f(z)}\right| \leq \sin^{-1}(N_2). \tag{32}$$

Inequality (??) together with (??) and (??) show that

$$\begin{aligned}\left|\arg\frac{zs_n'(z)}{s_n(z)}\right| &\leq \left|\arg\frac{s_n'(z)}{f'(z)}\right| + \left|\arg\frac{zf'(z)}{f(z)}\right| + \left|\arg\frac{f(z)}{s_n(z)}\right| \\ &< \sin^{-1}(K_1) + \sin^{-1}\left(\frac{2}{7}\right) + \sin^{-1}(N_2),\end{aligned}$$

for $|z| < 1/2$. Finally, we see that

$$\left|\arg\frac{zs_n'(z)}{s_n(z)}\right| < \frac{\pi}{2}$$

whenever

$$\sin^{-1}(K_1) + \sin^{-1}\left(\frac{2}{7}\right) + \sin^{-1}(N_2) \leq \frac{\pi}{2}.$$

However, the last inequality is easily seen to be true for all $n \geq 11$. Thus, $s_n(z)$ is starlike for $|z| < 1/2$ for $n \geq 11$. □

In [?], Theorem **??** was shown to be true for $n \geq 12$. As with Theorem **??**, a general version of Theorem **??** may now be stated.

Theorem 7. *Let $f \in \mathcal{G}(\alpha)$ for some $\alpha \in (0,1)$. Then, the section $s_n(z)$ of f is starlike in the disk $|z| < 1/2$ for $n \geq N$, where N is the smallest natural number such that*

$$\sin^{-1}(K_\alpha) + \sin^{-1}\left(\frac{2\alpha}{4\alpha+3}\right) + \sin^{-1}(N_\alpha) \leq \frac{\pi}{2},$$

where K_α is given by (??) and

$$N_\alpha = \frac{1}{2^n}\left(\frac{\alpha}{n(n+1)} + R(\alpha)\right)$$

*with $R(\alpha)$ as in Lemma **??**(2). Here the natural number N depends on the values of α.*

Proof. Following the proof of Theorem **??**, let $f \in \mathcal{G}(\alpha)$. Then, Case 2 of Lemma **??** gives that

$$\left|\frac{s_n(z)}{f(z)} - 1\right| < N_\alpha \quad \text{for } |z| \leq 1/2$$

which together with (??) and (??) easily imply that

$$\left|\arg\frac{zs_n'(z)}{s_n(z)}\right| < \sin^{-1}(K_\alpha) + \sin^{-1}\left(\frac{2\alpha}{4\alpha+3}\right) + \sin^{-1}(N_\alpha),$$

for $|z| < 1/2$. The desired conclusion follows easily from the hypothesis. □

We end the paper with the following remark. For

$$\alpha = 5/6, 4/5, 3/4, 2/3, 1/2, 1/3, 1/4, 1/5, 1/6,$$

the corresponding values of $N = N(\alpha)$ in Theorem **??** obtained via Mathematica are given by

$$N(\alpha) = 10, 10, 9, 9, 7, 6, 5, 4, 4,$$

respectively. We observe that these values are exactly the same as the corresponding values obtained for Theorem **??**.

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