Ramp Up Mathematics

–Calculus–

Exercises and Sample Solutions

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Exercises and Sample Solutions

Version as of 03-Jun-2026

1 Exercise

(a) Prove the following statement: The absolute value $|\cdot| : \mathbb{R} \to \mathbb{R}_0^+$ defines a norm on the real vector space $\mathbb{R}$.

(b) Let $0 < q < 1 < \beta$. Are the following sequences $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$, $(c_n)_{n=0}^{\infty}$ with $a_n := q^n$, $b_n := \sqrt[n]{\beta}$, and $c_n := \sqrt[n]{n}$ convergent? If so, what is their limit and how do we prove the convergence?

(c) Are the sequences $(d_n)_{n=0}^{\infty}$ and $(e_n)_{n=0}^{\infty}$, with

$$d_n = \frac{2^n - n^{10} + 5n^2}{3^n + n^5 + 4}, \quad e_n = \frac{3^n + n! - 5n^5}{4^n + n^7} \tag{1.1}$$

convergent and if so, what is their limit?

Solution of Exercise 1(a)

Recall the definition of the absolute value

$$|\cdot| : \mathbb{R} \to \mathbb{R}_0^+ \qquad x \mapsto |x| := \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases} \tag{1.2}$$

We verify the three norm axioms.

Positivity and definiteness: For all $x \in \mathbb{R}$, we have $|x| \geq 0$. Moreover, $|x| = 0$ if and only if $x = 0$.

Homogeneity: Let $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. Then

$$|\lambda x| = |\lambda|\,|x|. \tag{1.3}$$

Triangle inequality: Let $x, y \in \mathbb{R}$. We distinguish the cases $xy \geq 0$ and $xy < 0$.

If $xy \geq 0$, then x and y have the same sign, hence

$$|x + y| = |x| + |y|. \tag{1.4}$$

If $xy < 0$, then x and y have opposite signs. Without loss of generality, assume $|x| \geq |y|$. Then

$$|x + y| = ||x| - |y|| \leq |x| \leq |x| + |y|. \tag{1.5}$$

In both cases $xy \geq 0$ and $xy < 0$, the triangle inequality

$$|x + y| \;\leq\; |x| + |y| \tag{1.6}$$

holds true.

Since all three norm axioms are satisfied, the absolute value defines a norm on $\mathbb{R}$.

Solution of Exercise 1(b)

First we investigate $\boldsymbol{(a_n)_{n=0}^{\infty}}$ for fixed $q \in (0, 1)$. Since $0 < q < 1$, we have

$$a_{n+1} \;=\; q^{n+1} \;=\; q \cdot q^n \;<\; q^n \;=\; a_n, \quad \text{for all } n \in \mathbb{N}. \tag{1.7}$$

Hence the sequence $(a_n)_{n=0}^{\infty}$ is monotonically decreasing. Moreover, $q^n > 0$ for all n, so the sequence $(a_n)_{n=0}^{\infty}$ is bounded below by 0. It follows that $(a_n)_{n=0}^{\infty}$ is convergent. Let

$$a \;:=\; \lim_{n\to\infty} a_n \;\geq\; 0. \tag{1.8}$$

Taking limits in the identity $a_{n+1} = q^{n+1} = q\,q^n = a_n$, we obtain

$$a \;=\; \lim_{n\to\infty} \{a_{n+1}\} \;=\; q \lim_{n\to\infty} \{a_n\} \;=\; q\,a. \tag{1.9}$$

Hence, $(1 - q)a = 0$, and since $1 - q \neq 0$ this implies

$$a \;=\; 0. \tag{1.10}$$

Now consider $\boldsymbol{(b_n)_{n=0}^{\infty}}$ with $b_n = \sqrt[n]{\beta}$. First, we observe that

$$b_n \;=\; \sqrt[n]{\beta} > 1 \quad \text{for all } n \in \mathbb{N}. \tag{1.11}$$

Next, we observe that the sequence (b_n) is strictly decreasing, since

$$\frac{b_n}{b_{n+1}} \;=\; \frac{\beta^{\frac{1}{n}}}{\beta^{\frac{1}{n+1}}} \;=\; \beta^{\frac{1}{n}-\frac{1}{n+1}} \;=\; \beta^{\frac{1}{n(n+1)}} \;=\; b_{n(n+1)} \;>\; 1. \tag{1.12}$$

As (b_n) is decreasing and bounded below by 1, it converges to the limit

$$b \;:=\; \inf_{n\in\mathbb{N}} b_n \;\geq\; 1. \tag{1.13}$$

To show that $b = 1$, we take limits in (1.12) and obtain

$$b \;=\; \lim_{n\to\infty} \{b_{n(n+1)}\} \;=\; \lim_{n\to\infty} \left\{ \frac{b_n}{b_{n+1}} \right\} \;=\; \frac{\lim_{n\to\infty}\{b_n\}}{\lim_{n\to\infty}\{b_{n+1}\}} \;=\; \frac{b}{b} \;=\; 1. \quad (1.14)$$

Finally we analyze $\boldsymbol{(c_n)_{n=0}^{\infty}}$ with $c_n = \sqrt[n]{n}$. As for b_n, we have that

$$c_n \;=\; \sqrt[n]{n} \;>\; 1, \quad \text{for all } n \in \mathbb{N}. \quad (1.15)$$

Let $\varepsilon > 0$. According to the lecture,

$$\lim_{n\to\infty} \left\{ \frac{n}{(1+\varepsilon)^n} \right\} \;=\; 0\,, \quad (1.16)$$

and therefore there exists $n_0 \in \mathbb{N}$ such that

$$\forall\, n \geq n_0 : \qquad \frac{n}{(1+\varepsilon)^n} \;\leq\; 1\,, \quad (1.17)$$

and, hence, also

$$\forall\, n \geq n_0 : \qquad \sqrt[n]{\frac{n}{(1+\varepsilon)^n}} \;\leq\; 1\,. \quad (1.18)$$

So, for all $n \geq n_0$, we have that

$$1-\varepsilon \;\leq\; 1 \;\;\leq\;\; c_n \;=\; (1+\varepsilon)\,\sqrt[n]{\frac{n}{(1+\varepsilon)^n}} \;\leq\; 1+\varepsilon\,. \quad (1.19)$$

In summary,

$$\forall\, \varepsilon > 0\; \exists\, n_0 \in \mathbb{N}\; \forall\, n \geq n_0 : \quad |c_n - 1| \;\leq\; \varepsilon\,, \quad (1.20)$$

which means that $c_n \to 1$, as $n \to \infty$.

Solution of Exercise 1(c)

For the analyis of $d-n$, we first factor out the leading term of the numerator and the denominator, respectively,

$$d_n \;=\; \frac{2^n - n^{10} + 5n^2}{3^n + n^5 + 4} \;=\; \left(\frac{2}{3}\right)^n \frac{1 - n^{10}2^{-n} + 5n^2 2^{-n}}{1 + n^5 3^{-n} + 4(3^{-n})}\,. \quad (1.21)$$

According to the lecture and Exercise 1(b),

$$\begin{aligned}\lim_{n\to\infty}\left\{\left(\frac{2}{3}\right)^n\right\} &= \lim_{n\to\infty}\left\{n^{10}2^{-n}\right\} = \lim_{n\to\infty}\left\{n^{2}2^{-n}\right\} = \lim_{n\to\infty}\left\{n^{5}3^{-n}\right\} \\ &= \lim_{n\to\infty}\left\{3^{-n}\right\} = 0\,. \end{aligned} \tag{1.22}$$

Hence

$$\begin{aligned}\lim_{n\to\infty}\{d_n\} &= \lim_{n\to\infty}\left\{\left(\frac{2}{3}\right)^n\right\}\lim_{n\to\infty}\left\{\frac{1-n^{10}2^{-n}+5n^2 2^{-n}}{1+n^5 3^{-n}+4(3^{-n})}\right\} \\ &= 0\cdot\frac{1-0+0}{1+0+0} = 0\,. \end{aligned} \tag{1.23}$$

Similarly, to investigate e_n, we first factor out the leading term of the numerator and the denominator, respectively,

$$e_n = \frac{3^n+n!-5n^5}{4^n+n^7} = \frac{n!}{4^n}\,\frac{1+(3^n/n!)-(5n^5/n!)}{1+n^7 4^{-n}}\,. \tag{1.24}$$

Note that, for even $n = 2k \in \mathbb{N}$,

$$\begin{aligned} n! &= 2k(2k-1)(2k-2)\cdots(k+1)k(k-1)\cdots 3\cdot 2\cdot 1 \\ &\geq 2k(2k-1)(2k-2)\cdots(k+1) \geq k^k \geq \left(\frac{n}{2}\right)^{\frac{n}{2}}, \end{aligned} \tag{1.25}$$

and for odd $n = 2k-1 \in \mathbb{N}$,

$$\begin{aligned} n! &= (2k-1)(2k-2)\cdots(k+1)k(k-1)\cdots 3\cdot 2\cdot 1 \\ &\geq (2k-1)(2k-2)\cdots(k+1)k \geq k^k \geq \left(\frac{n+1}{2}\right)^{\frac{n+1}{2}}. \end{aligned} \tag{1.26}$$

It follows that

$$\forall\, n\in\mathbb{N}: \qquad n! \geq \left(\frac{n}{2}\right)^{\frac{n}{2}}. \tag{1.27}$$

With this estimate we obtain, for $n \geq 256$, that

$$\frac{n!}{4^n} \geq \left(\frac{\sqrt{n}}{4\sqrt{2}}\right)^n \geq 2^n \to \infty, \quad n\to\infty. \tag{1.28}$$

and analogously

$$\lim_{n\to\infty}\left\{\frac{3^n}{n!}\right\} = \lim_{n\to\infty}\left\{\frac{n^5}{n!}\right\} = \lim_{n\to\infty}\left\{\frac{n^7}{4^n}\right\} = 0\,, \tag{1.29}$$

and finally

$$e_n \to \infty, \quad n\to\infty. \tag{1.30}$$

2 Exercise

Let $N \in \mathbb{N}$.

(a) Prove that the maps

$$\begin{aligned} \|\cdot\|_1 : \mathbb{R}^N \to \mathbb{R}_0^+, \quad x \;=\; (x_1,\dots,x_N) \mapsto \sum_{i=1}^N |x_i|, \\ \|\cdot\|_\infty : \mathbb{R}^N \to \mathbb{R}_0^+, \quad x \;=\; (x_1,\dots,x_N) \mapsto \max_{1\le i\le N} |x_i| \end{aligned} \tag{2.1}$$

define norms on the vector space $\mathbb{R}^N$.

(b) Prove the following: The norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $\mathbb{R}^N$ are equivalent. That is, there exist constants $0 < c \le C < \infty$ such that, for all $x \in \mathbb{R}^N$,

$$c\|x\|_\infty \;\le\; \|x\|_1 \;\le\; C\|x\|_\infty. \tag{2.2}$$

(c) Are the sequences

$$a_n \;=\; \begin{pmatrix} 2^{-n} \\ \frac{n^2}{n^2+1} \\ \sqrt[n]{5} \end{pmatrix}, \quad b_n \;=\; \begin{pmatrix} (\frac{2}{e})^n \\ \sqrt[n]{n!} \\ \frac{n+2}{3n+5} \end{pmatrix} \tag{2.3}$$

convergent in $\mathbb{R}^3$?

Solution of Exercise 2(a)

The ℓ^1-Norm We verify the norm axioms.

Positivity and definiteness: For all $x \in \mathbb{R}^N$, we have $|x_i| \ge 0$, for each i, hence

$$\|x\|_1 \;=\; \sum_{i=1}^N |x_i| \;\ge\; 0. \tag{2.4}$$

Moreover, $\|x\|_1 = 0$ implies $|x_i| = 0$ for all i, and thus $x = 0$. Conversely, $\|0\|_1 = 0$.

Homogeneity: Let $x \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$. Then

$$\|\lambda x\|_1 \;=\; \sum_{i=1}^N |\lambda x_i| \;=\; |\lambda| \sum_{i=1}^N |x_i| \;=\; |\lambda|\,\|x\|_1. \tag{2.5}$$

Triangle inequality. Let $x, y \in \mathbb{R}^N$. Using the triangle inequality in $\mathbb{R}$, we obtain

$$\|x+y\|_1 \;=\; \sum_{i=1}^{N} |x_i + y_i| \;\le\; \sum_{i=1}^{N} \big\{|x_i| + |y_i|\big\} \;=\; \|x\|_1 + \|y\|_1. \tag{2.6}$$

Thus, $\|\cdot\|_1$ is a norm on $\mathbb{R}^N$.

The ℓ^∞-Norm Again, we verify the norm axioms.

Positivity and definiteness: For all $x \in \mathbb{R}^N$, we have $|x_i| \ge 0$ for each i, hence

$$\|x\|_\infty \;=\; \max_{1\le i\le N} |x_i| \;\ge\; 0. \tag{2.7}$$

Moreover, $\|x\|_\infty = 0$ implies $|x_i| = 0$ for all i, and therefore $x = 0$. Conversely, $\|0\|_\infty = 0$.

Homogeneity: Let $x \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$. Then

$$\|\lambda x\|_\infty \;=\; \max_{1\le i\le N} |\lambda x_i| \;=\; |\lambda| \max_{1\le i\le N} |x_i| \;=\; |\lambda|\, \|x\|_\infty. \tag{2.8}$$

Triangle inequality. Let $x, y \in \mathbb{R}^N$. For each $i = 1, \ldots, N$, the triangle inequality in $\mathbb{R}$ yields

$$|x_i + y_i| \;\le\; |x_i| + |y_i|. \tag{2.9}$$

Taking the maximum over all indices, we obtain

$$\|x+y\|_\infty \;=\; \max_{1\le i\le N} |x_i + y_i| \;\le\; \max_{1\le i\le N} (|x_i| + |y_i|) \;\le\; \|x\|_\infty + \|y\|_\infty. \tag{2.10}$$

Thus, $\|\cdot\|_\infty$ is a norm on $\mathbb{R}^N$.

Solution of Exercise 2(b)

Let $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$.

Recall the definitions

$$\|x\|_1 \;:=\; \sum_{i=1}^{N} |x_i|, \qquad \|x\|_\infty \;:=\; \max_{1\le i\le N} |x_i|. \tag{2.11}$$

Upper bound: For each $i = 1, \ldots, N$, we have $|x_i| \le \|x\|_\infty$. Hence,

$$\|x\|_1 \;=\; \sum_{i=1}^{N} |x_i| \;\le\; \sum_{i=1}^{N} \|x\|_\infty \;=\; N\,\|x\|_\infty. \tag{2.12}$$

Lower bound: Let $j \in \{1, \ldots, N\}$ such that $|x_j| = \|x\|_\infty$. Then,

$$\|x\|_1 \;=\; \sum_{i=1}^{N} |x_i| \;\ge\; |x_j| \;=\; \|x\|_\infty. \tag{2.13}$$

Combining both inequalities, we obtain

$$\|x\|_\infty \;\le\; \|x\|_1 \;\le\; N\,\|x\|_\infty, \tag{2.14}$$

which proves the equivalence of the two norms.

Solution of Exercise 2(c)

It is possible to prove that not only $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are equivalent on $\mathbb{R}^N$ but that *all* norms on a finite dimensional vector space are equivalent. The same is, however, false for infinite dimensional vector spaces.

For $N = 3$ the equivalence of all norms on $\mathbb{R}^3$ means that the convergence of sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ of vectors $a_n, b_n \in \mathbb{R}^3$ does not depend on the norm we are using. Using the $\|\cdot\|_\infty$-norm, we see that the convergence $a_n \to a = (a_1, a_2, a_3)$

$$\forall\, \varepsilon > 0\; \exists\, n_0 \in \mathbb{N}\; \forall\, n \ge n_0 : \quad \|a_n - a\|_\infty \;=\; \max_{i=1,2,3} |a_{n,i} - a_i| \;\le\; \varepsilon \tag{2.15}$$

is equivalent to the convergence of each component $a_{n,1} = 2^{-n}$, $a_{n,2} = \frac{n^2}{n^2+1}$, and $a_{n,3} = \sqrt[n]{5}$ of the sequence. By using methods from Exercise 1, we see that $a_{n,1} \to 0$, $a_{n,2} \to 1$, and $a_{n,3} \to 1$, as $n \to \infty$. Hence

$$a_n \;\to\; a \;=\; \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \;\; n \to \infty. \tag{2.16}$$

Regarding b_n we notice that $e \approx 2.7 > 2$ and hence $(\frac{2}{e})^n \to 0$, as $n \to \infty$. Since $n! \ge (\frac{n}{2})^{n/2}$, however, we have

$$\sqrt[n]{n!} \;\ge\; \sqrt[n]{(\tfrac{n}{2})^{n/2}} \;=\; \sqrt{\frac{n}{2}} \;\to\; \infty\,, \tag{2.17}$$

so the second component of b_n diverges. Consequently, the vector b_n diverges.

3 Exercise

Prove the following assertions:

(a) The map $\langle \cdot, \cdot \rangle : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$\langle x, y \rangle \ := \ \sum_{i=1}^{N} x_i y_i, \quad x \ = \ (x_1, \dots, x_N), \ y \ = \ (y_1, \dots, y_N), \tag{3.1}$$

is an inner (scalar) product on $\mathbb{R}^N$.

(b) Let $(V, \langle \cdot, \cdot \rangle)$ be a real inner product space. Then the map $\| \cdot \| : V \to \mathbb{R}$ defined by

$$\|x\| \ := \ \sqrt{\langle x, x \rangle} \tag{3.2}$$

is a norm on V.

Solution of Exercise 3(a)

We verify the axioms of a scalar (inner) product.

Bilinearity: Let $x, y, w, z \in \mathbb{R}^N$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} \langle x + \alpha y, w + \beta z \rangle &= \sum_{i=1}^{N} (x_i + \alpha y_i)(w_i + \beta z_i) \\ &= \sum_{i=1}^{N} x_i w_i + \beta \sum_{i=1}^{N} x_i z_i + \alpha \sum_{i=1}^{N} y_i w_i + \alpha\beta \sum_{i=1}^{N} y_i z_i \\ &= \langle x, w \rangle + \beta \langle x, z \rangle + \alpha \langle y, w \rangle + \alpha\beta \langle y, z \rangle. \end{aligned} \tag{3.3}$$

Symmetry: For all $x, y \in \mathbb{R}^N$, we have

$$\langle x, y \rangle \ = \ \sum_{i=1}^{N} x_i y_i \ = \ \sum_{i=1}^{N} y_i x_i \ = \ \langle y, x \rangle. \tag{3.4}$$

Positive definiteness: For all $x \in \mathbb{R}^N$, we have

$$\langle x, x \rangle \ = \ \sum_{i=1}^{N} x_i^2 \ \geq \ 0, \tag{3.5}$$

with equality if, and only if, $x_1^2 = x_2^2 = \cdots = x_N^2 = 0$ which is equivalent to $x = 0$.

Since symmetry, linearity in the first argument, and positive definiteness are satisfied, the Euclidean scalar product defines an inner product on $\mathbb{R}^N$.

Solution of Exercise 3(b)

We verify the norm axioms.

Positivity and definiteness: By positive definiteness of the inner product,

$$\langle x, x \rangle \;\geq\; 0, \quad \text{for all } x \in V. \tag{3.6}$$

Hence $\|x\| \geq 0$. Moreover,

$$\|x\| \;=\; 0 \;\;\Leftrightarrow\;\; \langle x, x \rangle \;=\; 0 \;\;\Leftrightarrow\;\; x \;=\; 0. \tag{3.7}$$

Homogeneity: Let $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. Then

$$|\lambda x| \;=\; |\lambda|\,|x|. \tag{3.8}$$

Triangle inequality: Let $x, y \in V$. Using bilinearity and symmetry, we compute

$$\|x+y\|^2 \;=\; \langle x+y,\, x+y \rangle \;=\; \|x\|^2 + 2\langle x, y \rangle + \|y\|^2. \tag{3.9}$$

By the Cauchy–Schwarz inequality, we have

$$|\langle x, y \rangle| \;\leq\; \|x\|\,\|y\|, \tag{3.10}$$

and hence

$$\|x+y\|^2 \;\leq\; \big(\|x\| + \|y\|\big)^2. \tag{3.11}$$

Taking square roots gives

$$\|x+y\| \;\leq\; \|x\| + \|y\|. \tag{3.12}$$

Thus, all norm axioms are satisfied, and $\|\cdot\|$ is a norm on V.

4 Exercise

(a) Prove that $f : \mathbb{R} \to \mathbb{R},\ x \mapsto x^2$ is continuous on $\mathbb{R}$.

(b) Prove that

$$g : \mathbb{R} \ \to \ \mathbb{R}, \quad x \ \mapsto \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{4.1}$$

is not continuous at $x_0 = 0$.

(c) Which of the following functions $h_1, h_2, h_3, h_4 : \mathbb{R} \to \mathbb{R}$ are continuous on $\mathbb{R}$?

$$h_1(x) \ = |x|, \quad h_2(x) \ = \begin{cases} \frac{x}{|x|}, & x \neq 0, \\ 0, & x = 0, \end{cases} \tag{4.2}$$

$$h_3(x) \ = \begin{cases} \frac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases} \quad h_4(x) \ = \begin{cases} \frac{x^2-4}{x+2}, & x \neq -2, \\ -4, & x = -2. \end{cases} \tag{4.3}$$

Solution of Exercise 4(a)

For this, we need to prove

$$\lim_{x \to x_0} f(x) \ = \ f(x_0), \tag{4.4}$$

i.e., $\lim_{x \to x_0} x^2 = x_0^2$, for all $x_0 \in \mathbb{R}$. Let $x_0 \in \mathbb{R}$ be arbitrary and let $(x_n)_{n=1}^{\infty} \in (\mathbb{R} \backslash \{x_0\})^{\mathbb{N}}$ be an arbitrary sequence for which $\lim_{n \to \infty} x_n = x_0$. From the lecture we know that, if $a_n \to a$ and $b_n \to b$, then also $a_n \cdot b_n \to a \cdot b$. Hence

$$\lim_{n \to \infty} \left\{x_n^2\right\} \ = \ \lim_{n \to \infty} \left\{x_n \cdot x_n\right\} \ = \ \left(\lim_{n \to \infty} \{x_n\}\right) \cdot \left(\lim_{n \to \infty} \{x_n\}\right) \ = \ x_0 \cdot x_0 \ = \ x_0^2. \tag{4.5}$$

Since x_n was arbitrary, this proves $\lim_{x \to x_0}\{x^2\} = x_0^2$, i.e., $\lim_{x \to x_0}\{f(x)\} = f(x_0)$. Hence, f is continuous at x_0. As x_0 was arbitrary, f is continuous on $\mathbb{R}$.

Solution of Exercise 4(b)

If $x_n = -1/n$, then $x_n \to 0$, as $n \to \infty$, but

$$\lim_{n \to \infty} \{g(x_n)\} \ = \ \lim_{n \to \infty} \{0\} \ = \ 0 \ \neq \ 1 \ = \ g(0). \tag{4.6}$$

Hence, g is not continuous at $x_0 = 0$.

Solution of Exercise 4(c)

h_1 and h_4 are continuous on $\mathbb{R}$. h_2 and h_3 are both not continuous at $x_0 = 0$ and, therefore, also not continuous on $\mathbb{R}$.

5 Exercise

(a) Prove that $f : \mathbb{R} \to \mathbb{R},\ x \mapsto x^2$ is differentiable on $\mathbb{R}$.

(b) Prove that $h_1(x) = |x|$ is not differentiable at $x = 0$.

(c) Let $F : \mathbb{R} \to \mathbb{R}$, $x \mapsto x \cdot \exp[-x^2]$. Determine all local and global maxima and minima of F and sketch the graph of F.

Solution of Exercise 5(a)

Let $x_0 \in \mathbb{R}$ be arbitrary and $(x_n)_{n=1}^{\infty} \in (\mathbb{R} \setminus \{0\})^{\mathbb{N}}$ be a convergent sequence with $\lim_{n\to\infty} x_n = x_0$. Then, we have

$$\lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = \lim_{n\to\infty} \frac{x_n^2 - x_0^2}{x_n - x_0} = \lim_{n\to\infty} \frac{(x_n + x_0) \cdot (x_n - x_0)}{x_n - x_0}$$
$$= \lim_{n\to\infty} (x_n + x_0) = 2x_0 = f'(x_0). \tag{5.1}$$

Therefore, f is differentiable at x_0, and since $x_0 \in \mathbb{R}$ is arbitrary, f is differentiable on $\mathbb{R}$.

Solution of Exercise 5(b)

Let $x_n = (-1)^n/n$, then $x_n \neq 0$, for all n, but $x_n \to 0$, as $n \to \infty$. Observe that $h_1(0) = 0$. Hence, the difference quotient for this sequence equals

$$\frac{h_1(x_n) - h_1(0)}{x_n - 0} = \frac{|(-1)^n\, n^{-1}|}{(-1)^n\, n^{-1}} = (-1)^n\,, \tag{5.2}$$

which is divergent. It follows that h_1 is not differentiable at 0.

Solution of Exercise 5(c)

To begin with, notice that $F(x) > 0$ for $x > 0$, $F(x) < 0$ for $x < 0$ and $F(x) = 0$ for $x = 0$. Secondly, for $x \to \pm\infty$ we have $F(x) \to 0$ due to the exponential decrease of $\exp[-x^2]$. Now, we determine the local extrema. For this, we need the derivative of $F(x) = x \cdot \exp[-x^2]$. We already now the derivatives of x, x^2 and $\exp[x]$, but how do we compute the derivative of F=? We can write F as a product $F(x) = u(x) \cdot v(x)$ with $u(x) = x$ and $v(x) = \exp[-x^2]$. We already now that $u'(x) = 1$. From Leibniz' rule it follows that

$$\begin{aligned} F'(x) &= \big(u(x) \cdot v(x)\big)' = u'(x) \cdot v(x) + u(x) \cdot v'(x) \\ &= 1 \cdot \exp[-x^2] + x \cdot \big(\exp[-x^2]\big)'. \end{aligned} \tag{5.3}$$

$\exp[-x^2]$ can be written as $v_1[v_2(x)]$ with $v_1[x] = \exp[x]$ and $v_2(x) = -x^2$. Here, we know that $v_1'[x] = \exp[x] = v_1[x]$ and $v_2'(x) = -2x$. The chain rule now implies that

$$\big(\exp[-x^2]\big)' = v_1'[v_2(x)] \cdot v_2'(x) = \exp[v_2(x)] \cdot (-2x) = -2x \exp[-x^2]. \tag{5.4}$$

Putting these results together, we arrive at

$$F'(x) = (1 - 2x^2) \cdot \exp[-x^2]. \tag{5.5}$$

Note that F is differentiable on $\mathbb{R}$ and vanishes, as $x \to \pm\infty$. Moreover, F is strictly positive on $\mathbb{R}^+ = (0, \infty)$ and strictly negative on $\mathbb{R}^- = (-\infty, 0)$. It follows that all local minima occur at points in $\mathbb{R}^-$, all local maxima occur at points in $\mathbb{R}^+$, and at these extremal points the derivative F' of F vanishes,

$$F'(x) = 0 \;\Leftrightarrow\; (1 - 2x^2) \exp[-x^2] = 0 \;\Leftrightarrow\; 1 - 2x^2 = 0 \;\Leftrightarrow\; x = \pm\frac{1}{\sqrt{2}}. \tag{5.6}$$

Without computing the second derivative we may already conclude at this point that the local and in fact global minimum of F occurs at $x_- = -\frac{1}{\sqrt{2}}$ and the local and in fact global maximum of F occurs at $x_+ = \frac{1}{\sqrt{2}}$.

We nevertheless compute the second derivative of F, obtain $F''(x) = (4x^3 - 6x) \exp[-x^2] = x(4x^2 - 6) \exp[-x^2]$, in general, and observe that

$$F''(x_\pm) = F''(\pm\tfrac{1}{\sqrt{2}}) = \mp 2\sqrt{\frac{2}{e}}. \tag{5.7}$$

So, indeed, $\big(\pm 1/\sqrt{2}, \pm \exp[-1/2]/\sqrt{2}\big)$ denotes the only local and global maximum/minimum.

6 Exercise

(a) Let $M, N \in \mathbb{N}$, $A \in \mathbb{R}^{M\times N}$ and $f : \mathbb{R}^N \to \mathbb{R}^M,\ x \mapsto A \cdot x$. Prove that f is totally differentiable on $\mathbb{R}^N$ and compute its derivative $(Df)(x_0)$ for all $x_0 \in \mathbb{R}^N$.

(b) Compute also the Jacobian $J_f(x_0)$ for all $x_0 \in \mathbb{R}^N$.

(c) Compute the second order Taylor polynomial

$$T_2[g, x_0; x_0 + z] \;=\; \sum_{p=0}^{2} \frac{1}{p!} \sum_{j_1,\ldots,j_p=1}^{N} \left[\frac{\partial^p g(x_0)}{\partial x_{j_1} \ldots \partial x_{j_p}}\right] \cdot z_{j_1} \cdot \ldots \cdot z_{j_p} \tag{6.1}$$

of the function $g : \mathbb{R}^2 \to \mathbb{R}$, $(x_1, x_2) \mapsto (x_1^2 + x_2^2) \cdot \exp[-x_1^2 - x_2^2]$ at $x_0 = (0, 0)$.

Solution of Exercise 6(a)

Let $x_0 \in \mathbb{R}^N$ be given. We wish to find a linear function $f'(x_0) : \mathbb{R}^N \to \mathbb{R}^M$ such that

$$\lim_{z\to 0} \frac{\big|f(x_0 + z) - \{f(x_0) + f'(x_0) \cdot z\}\big|}{|z|} \;=\; 0. \tag{6.2}$$

For this, we observe that the difference

$$f(x_0 + z) - f(x_0) \;=\; A \cdot (x_0 + z) - Ax_0 \;=\; Ax_0 + Az - Ax_0 \;=\; Az. \tag{6.3}$$

Therefore, choosing $f'(x_0) = A$ we obtain

$$f(x_0 + z) - \{f(x_0) + f'(x_0) \cdot z\} \;=\; 0, \tag{6.4}$$

which implies the total differentiability of f. We notice that $f'(x_0) = A$ is independent of x_0. (In case that $M = N = 1$ we already knew before that for $f(x) = ax$ we have that $f'(x) = a$ is a constant.)

Solution of Exercise 6(b)

We now study the function f componentwise. We have

$$f(x) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \tag{6.5}$$

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N \\ \vdots \\ a_{M1}x_1 + a_{M2}x_2 + \ldots + a_{MN}x_N \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{N} a_{1j}\, x_j \\ \sum_{j=1}^{N} a_{2j}\, x_j \\ \vdots \\ \sum_{j=1}^{N} a_{Mj}\, x_j \end{pmatrix}.$$

Hence, if $i \in \mathbb{Z}_1^M$ and $j \in \mathbb{Z}_1^N$, the partial derivative of the i^{th} component f_i of f w.r.t. x_j is given by

$$\frac{\partial f_i(x)}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\sum_{k=1}^{N} a_{ik}\, x_k\right] = \sum_{k=1}^{N} a_{ik}\frac{\partial x_k}{\partial x_j} = \sum_{k=1}^{N} a_{ik}\delta_{jk} = a_{ij}. \tag{6.6}$$

This gives

$$J_f(x_0) = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_N} \\ \vdots & & \vdots \\ \frac{\partial f_M(x)}{\partial x_1} & \cdots & \frac{\partial f_M(x)}{\partial x_N} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & \vdots \\ a_{M1} & \cdots & a_{MN} \end{pmatrix} = A. \tag{6.7}$$

in accordance with Exercise 6(a).

Solution of Exercise 6(c)

Obviously, g is a smooth function. We compute its first and second partial derivatives.

$$\frac{\partial g(x_1,x_2)}{\partial x_1} = \exp[-x_1^2 - x_2^2] \cdot 2x_1 \cdot (1 - x_1^2 - x_2^2), \tag{6.8}$$

$$\frac{\partial g(x_1,x_2)}{\partial x_2} = \exp[-x_1^2 - x_2^2] \cdot 2x_2 \cdot (1 - x_1^2 - x_2^2), \tag{6.9}$$

$$\begin{aligned}\frac{\partial^2 g(x_1,x_2)}{\partial x_1^2} &= 2\exp[-x_1^2 - x_2^2](1 - x_1^2 - x_2^2 - 2x_1^2 - 2x_1^2 + 2x_1^4 + 2x_1^2x_2^2) \\ &= 2\exp[-x_1^2 - x_2^2](1 - 5x_1^2 - x_2^2 + 2x_1^4 + 2x_1^2x_2^2),\end{aligned} \tag{6.10}$$

$$\frac{\partial^2 g(x_1,x_2)}{\partial x_1\,\partial x_2} = \frac{\partial^2 g(x_1,x_2)}{\partial x_2\,\partial x_1} = -4x_1x_2\,\exp[-x_1^2 - x_2^2]\,(2 - x_1^2 - x_2^2), \tag{6.11}$$

$$\frac{\partial^2 g(x_1,x_2)}{\partial x_2^2} = 2\exp[-x_1^2 - x_2^2]\,(1 - x_1^2 - 5x_2^2 + 2x_2^4 + 2x_1^2x_2^2), \tag{6.12}$$

and hence

$$g(0,0) = 0 \tag{6.13}$$

$$\frac{\partial g(0,0)}{\partial x_1} = 0 \tag{6.14}$$

$$\frac{\partial g(0,0)}{\partial x_2} = 0 \tag{6.15}$$

$$\frac{\partial^2 g(0,0)}{\partial x_1^2} = 2 \tag{6.16}$$

$$\frac{\partial^2 g(0,0)}{\partial x_1\,\partial x_2} = \frac{\partial^2 g(0,0)}{\partial x_2\,\partial x_1} = 0 \tag{6.17}$$

$$\frac{\partial^2 g(0,0)}{\partial x_2^2} = 2. \tag{6.18}$$

Therefore, we have

$$T_2[g,(0,0);z] = z_1^2 + z_2^2. \tag{6.19}$$