



## 8. Exercise Sheet 8

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### Exercise 8.1 ( 12 )

Let  $\mathfrak{h}$  be an infinite-dimensional separable complex Hilbert space and let  $\mathfrak{b} := \{\varphi_k\}_{k=1}^{\infty} \subseteq \mathfrak{h}$  be an ONB.

- (i) Show that  $\mathfrak{F}^{(n)}(\mathfrak{h})$  is also an infinite-dimensional separable complex Hilbert space, for every  $n \in \mathbb{N}$ , and that

$$\mathfrak{B}^{(n)} := \{\varphi_{k(1)} \otimes \cdots \otimes \varphi_{k(n)} \mid k(1), \dots, k(n) \in \mathbb{N}\} \quad (1)$$

is an ONB in  $\mathfrak{F}^{(n)}(\mathfrak{h})$ .

- (ii) Show that  $\mathfrak{F}(\mathfrak{h})$  is also an infinite-dimensional separable complex Hilbert space, and that, with  $\mathfrak{B}^{(0)} := \{\Omega\}$ ,

$$\mathfrak{B} := \bigcup_{n=0}^{\infty} \mathfrak{B}^{(n)} \quad (2)$$

is an ONB in  $\mathfrak{F}(\mathfrak{h})$ .

### Exercise 8.2 ( 6 )

Let  $\mathfrak{h}$  be a Hilbert space, let  $\varphi \in \mathfrak{h}$ , and let

$$(A^\dagger(\varphi), \mathcal{D}_{\sqrt{\mathcal{N}}}), (A(\varphi), \mathcal{D}_{\sqrt{\mathcal{N}}}) \in \mathfrak{L}[\mathfrak{F}(\mathfrak{h})]$$

be the corresponding creation and annihilation operators.

- (i) Show that  $(A^\dagger(\varphi), \mathcal{D}_{\sqrt{\mathcal{N}}})$  and  $(A(\varphi), \mathcal{D}_{\sqrt{\mathcal{N}}})$  are densely defined and relatively bounded with respect to  $\sqrt{\mathcal{N}}$ , and that

$$\langle A^\dagger(\varphi)[\Psi] | \Phi \rangle = \langle \Psi | A(\varphi)[\Phi] \rangle$$

holds for all  $\Psi, \Phi \in \mathcal{D}_{\sqrt{\mathcal{N}}}$ .

- (ii) Show that  $(A(\varphi), \mathcal{D}_{\sqrt{\mathcal{N}}})$  is closable and  $(A^\dagger(\varphi), \mathcal{D}_{\sqrt{\mathcal{N}}})$  is closed, and that  $(A^\dagger(\varphi), \mathcal{D}_{\sqrt{\mathcal{N}}})$  is the adjoint operator of  $(A(\varphi), \mathcal{D}_{\sqrt{\mathcal{N}}})$ .

- (iii) Show that  $(A^\dagger(\varphi) + A(\varphi), \mathcal{D}_{\sqrt{\mathcal{N}}}) \in \mathfrak{L}[\mathfrak{F}(\mathfrak{h})]$  is essentially self-adjoint.

Hint: This can be reduced to criterion (ii) in Theorem I.8, Stone's theorem for essentially self-adjoint operators. To this end, for  $\varepsilon \geq 0$  define  $K_\varepsilon := (1 + \varepsilon\sqrt{\mathcal{N}})^{-1/2}$  and  $A_\varepsilon := AK_\varepsilon$ ,  $A_\varepsilon^\dagger := K_\varepsilon A^\dagger$ , as well as  $F_\varepsilon := A_\varepsilon^\dagger + A_\varepsilon$ . Then  $A_\varepsilon^\dagger, A_\varepsilon, F_\varepsilon \in \mathfrak{L}(\mathfrak{h})$  are bounded and  $F_\varepsilon = F_\varepsilon^*$  is self-adjoint for all  $\varepsilon > 0$ . In particular,  $\text{Ran}(F_\varepsilon \pm i\lambda) = \mathfrak{F}(\mathfrak{h})$ , and  $F_\varepsilon \pm i\lambda$  is boundedly invertible with

$$\|(F_\varepsilon \pm i\lambda)^{-1}\|_{\mathcal{B}[\mathfrak{F}(\mathfrak{h})]} \leq \lambda^{-1}$$

for all  $\lambda > 0$ . Now show that

$$(F_0 \pm i\lambda)(F_\varepsilon \pm i\lambda)^{-1}\Phi \rightarrow \Phi$$

in the limit  $\varepsilon \rightarrow 0$ , for every  $\Phi \in \mathcal{D}_{\mathcal{N}}$ .

### Exercise 8.3 ( 6 )

Let  $\mathfrak{h}$  be an infinite-dimensional Hilbert space and let  $\mathfrak{b} := \{\varphi_k\}_{k=1}^{\infty} \subseteq \mathfrak{h}$  be an ONB.

- (i) Show that the fermionic creation and annihilation operators  $\{c^\dagger(\varphi), c(\varphi)\}_{\varphi \in \mathfrak{h}}$  satisfy the canonical anticommutation relations, or CAR,

$$\forall \varphi, \psi \in \mathfrak{h} : \{c^\dagger(\varphi), c^\dagger(\psi)\} = \{c(\psi), c(\varphi)\} = 0, \quad \{c(\psi), c^\dagger(\varphi)\} = \langle \psi | \varphi \rangle \cdot \mathbb{1}_{\mathfrak{F}_f(\mathfrak{h})}. \quad (3)$$

- (ii) Let  $\varphi \in \mathfrak{h}$ . Show that  $[c(\varphi)]^* = c^\dagger(\varphi)$  and that  $c(\varphi), c^\dagger(\varphi) \in \mathcal{B}[\mathfrak{F}_f(\mathfrak{h})]$  are bounded, and compute the corresponding operator norms.

- (iii) Using (i), (ii), and  $c(\varphi)\Omega = 0$ , show that

$$\mathfrak{B}_f := \{c^*(\varphi_{k(1)}) \cdots c^*(\varphi_{k(n)})\Omega \mid n \in \mathbb{N}, k(1), \dots, k(n) \in \mathbb{N}, k(1) < k(2) < \cdots < k(n)\} \quad (4)$$

is an ONB in  $\mathfrak{F}_f(\mathfrak{h})$ .