Technische Universität Braunschweig

7. Exercise Sheet 7

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Exercise 7.1 (12)

Let $A \in \mathbb{R}^{4\times 4}$ be the matrix

$$A = \begin{pmatrix} \frac{2}{5} & \frac{3}{10} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{10} & \frac{2}{5} & \frac{3}{10} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{5} & \frac{1}{10} & \frac{2}{5} \\ \frac{1}{5} & \frac{1}{10} & \frac{2}{5} & \frac{3}{10} \end{pmatrix}.$$

(i) Show that 1 is an eigenvalue of A, that it is the largest eigenvalue in modulus, and that it is simple.

(ii) Determine or estimate the second largest eigenvalue in modulus λ_2.

(ii) Let $q = |\lambda_2|$ and let

$$\pi = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

be the invariant distribution of A. Show that there exists $C > 0$, such that for any probability vector $\sigma \in (\mathbb{R}^+)^4$ satisfying

$$\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 1,$$

the following holds true

$$\|A^n\sigma - \pi\| \leq Cq^n.$$

Solutions.

(i) An explicit computation, using the fact that $\sum_{j=1}^4 A_{ij} = 1$, shows that $(1,1,1,1)$ is an eigenvector associated with the eigenvalue 1. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A, and let $0 \neq \phi \in \mathrm{Ker}(A-\lambda)$ be an eigenvector associated with λ. Let $0 \neq |\phi_i| = \max\{|\phi_j| : j \in \{1,\ldots,4\}\}$. Then one has

$$|\lambda||\phi_i| = |\lambda\phi_i| = \left|\sum_{j=1}^4 A_{ij}\phi_j\right| \leq \sum_{j=1}^4 A_{ij}\,|\phi_j| \leq |\phi_i|\sum_{j=1}^4 A_{ij} = |\phi_i|. \tag{1}$$

Equation (1) implies that $|\lambda| \leq 1$. Moreover, if $|\lambda| = 1$, then Eq. (1) implies that

$$\left|\sum_{j=1}^4 A_{ij}\phi_j\right| = \sum_{j=1}^4 A_{ij}\,|\phi_j|\,. \tag{2}$$

Hence, there exists $\theta \in [0, 2\pi)$ such that $\phi_j = |\phi_j|e^{i\theta}$. The second inequality in Eq. (1) implies that $A_{ij}|\phi_j| = |\phi_i|A_{ij}$, and hence $|\phi_j| = |\phi_i|$. Therefore, $\phi = z(1,1,1,1)$ for some $z \in \mathbb{C}$, which implies that $\lambda = 1$.

The above shows that 1 is the eigenvalue of A with largest modulus. Moreover, it also proves that if $\phi \in \mathrm{Ker}(A-1)$, then $\phi \in \langle(1,1,1,1)\rangle$. Thus,

$$\mathrm{Ker}(A-1) = \langle(1,1,1,1)\rangle.$$

Let us show that 1 is a simple eigenvalue. Define $V := \{\sigma \in \mathbb{C}^4 : \sum_{j=1}^4 \sigma_j = 0\}$. Note that

$$\mathbb{C}^4 = \mathrm{Ker}(A-1) \oplus V.$$

Moreover, V is A-invariant. Indeed, if $v \in V$, then

$$\sum_{j=1}^4 (Av)_j = \sum_{j=1}^4 \sum_{i=1}^4 A_{ji} v_i = \sum_{i=1}^4 v_i \sum_{j=1}^4 A_{ji} = \sum_{j=1}^4 v_j = 0. \tag{3}$$

One has $\mathrm{Ker}(A-1) = \mathrm{Ker}((A-1)^2)$. Indeed, take $\phi \in \mathrm{Ker}((A-1)^2)$ and write $\phi = v + w$, with $v \in \mathrm{Ker}(A-1)$ and $w \in V$. Since $v \in \mathrm{Ker}(A-1)$, one has

$$0 = (A-1)^2\phi = (A-1)^2 w.$$

Since V is A-invariant, $(A-1)w \in \mathrm{Ker}(A-1) \cap V = \{0\}$, and hence $w \in \mathrm{Ker}(A-1) \cap V = \{0\}$. Therefore, $\phi = v \in \mathrm{Ker}(A-1)$. Since

$$\mathrm{Ker}(A-1) = \mathrm{Ker}((A-1)^2),$$

it follows that 1 is a simple eigenvalue.

(ii) Let us take $\alpha = \{\pi, u_2, u_3, u_4\}$, with

$$u_2 = (1,0,-1,0), \qquad u_3 = (1,-1,1,-1), \qquad u_4 = (1,-1,-1,1).$$

Then one obtains

$$[A]_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & -\frac{2}{5} \\ \frac{1}{5} & 0 & \frac{1}{5} \\ 0 & \frac{1}{5} & \frac{1}{5} \end{pmatrix}. \tag{4}$$

Then the modulus of the second largest eigenvalue is given by

$$|\lambda_2| = \sup\{|\lambda| : \lambda \in \sigma(B)\} \le \|B^n\|_{op}^{1/n} \le \|B^n\|_\infty^{1/n}. \tag{5}$$

Taking $n = 2$, one obtains the estimate

$$q = |\lambda_2| \le \frac{\sqrt{2}}{5}.$$

(iii) First, we note that A is diagonalizable. Indeed, Eq. (4) gives that the characteristic polynomial of A is

$$p_A(z) = (z-1)\left(z^3 - \frac{1}{5}z^2 - \frac{1}{25}z + \frac{2}{125}\right) = (z-1)q(z).$$

One checks that the roots of q' are $1/5$ and $-1/15$, and none of them is a root of q. Therefore, p_A has simple roots, and hence A is diagonalizable. Let $1, \lambda_2, \lambda_3, \lambda_4$ be the eigenvalues of A. Consider $\theta := \{\pi, \phi_2, \phi_3, \phi_4\}$, a basis of eigenvectors of A. For $k \in \{2,3,4\}$, one has $\mathrm{Ker}(A - \lambda_k) \subset V$. Indeed, if $\phi \in \mathrm{Ker}(A - \lambda_k)$, then

$$\lambda_k \sum_{j=1}^4 \phi_j = \sum_{j=1}^4 (A\phi)_j = \sum_{j=1}^4 \sum_{i=1}^4 A_{j,i}\phi_i = \sum_{i=1}^4 \phi_i \sum_{j=1}^4 A_{j,i} = \sum_{i=1}^4 \phi_i. \tag{6}$$

Since $\lambda_k \neq 1$, it follows that $\phi \in V$. Therefore,

$$V = \bigoplus_{k=2}^{4} \mathrm{Ker}(A - \lambda_k). \tag{7}$$

If $\sigma \in (\mathbb{R}^+)^4$ satisfies $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 1$, then $\sigma - \pi \in V$. Hence, Eq. (7) implies that

$$\sigma - \pi = \sum_{j=2}^{4} \mu_j \phi_j, \qquad \mu_j \in \mathbb{C}. \tag{8}$$

Then one obtains

$$\|A^n \sigma - \pi\| = \|A^n(\sigma - \pi)\| \leq \sum_{j=2}^{4} |\mu_j||\lambda_j|^n \|\phi_j\| \leq C q^n, \tag{9}$$

where

$$C := \sup\{|\mu_1| + |\mu_2| + |\mu_3| : (\mu_1, \mu_2, \mu_3) \in P[I]_\beta^\theta(S)\} \sum_{j=2}^{4} \|\phi_j\|.$$

Here $S = \{\sigma \in [0,1]^4 : \sum_{j=1}^4 \sigma_j = 1\}$, $[I]_\beta^\theta$ is the change-of-basis matrix, with β denoting the standard basis, and $P : \mathbb{C}^4 \to \mathbb{C}^3$, $P(v_1, v_2, v_3, v_4) = (v_2, v_3, v_4)$.

Exercise 7.2 (6)

Let $\mathcal{H}$ be a Hilbert space. Let $(T, \mathcal{D})$ be a densely defined closed operator on $\mathcal{H}$, and let $W \geq 0$ be a finite-rank operator. For $\lambda \in \mathbb{R}$, define

$$T_\lambda := T + \lambda W. \tag{10}$$

For $\lambda \in \mathbb{R} \setminus \{0\}$ and $z \in \mathbb{C}$, consider the operator $(B(z,\lambda), \mathcal{D} \oplus \mathcal{H}) \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ given by

$$B(z,\lambda)(\phi \oplus \psi) = \big((T - z)\phi + W^{1/2}\psi\big) \oplus \big(W^{1/2}\phi - \lambda^{-1}\psi\big). \tag{11}$$

Let P denote the orthogonal projection onto the second component,

$$P : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}, \qquad P(\psi \oplus \phi) = 0 \oplus \phi,$$

and let $P^\perp := I - P$, that is,

$$P^\perp(\psi \oplus \phi) = \psi \oplus 0.$$

(i) Prove that the kernels of $B(z,\lambda)$ and $T_\lambda - z$ are isomorphic.

(ii) Assume that $z \notin \sigma(T)$. Show that the Feshbach-Schur map $F_P\big(B(z,\lambda)\big)$ exists, and compute it explicitly.

(iii) Deduce that, whenever $z \notin \sigma(T)$, $z \in \sigma_{\mathrm{p}}(T_\lambda)$ if and only if

$$-1 \in \sigma\Big(\lambda\, W^{1/2}(T - z)^{-1} W^{1/2}\Big). \tag{12}$$

Solution.

(i) Let us consider the linear transform

$$\Phi : \mathcal{D} \to \mathcal{D} \oplus \mathcal{H}, \qquad \Phi(\phi) = \phi \oplus \lambda W^{1/2}\phi.$$

If $\phi \in \operatorname{Ker}(T_\lambda - z)$, then $(T + \lambda W - z)\phi = 0$. Thus,

$$B(z,\lambda)\Phi(\phi) = (T - z)\phi + \lambda W^{1/2}W^{1/2}\phi \oplus 0 = 0, \tag{13}$$

that is, $\Phi(\phi) \in \operatorname{Ker}(B(z,\lambda))$. Conversely, if $\phi \oplus \psi \in \operatorname{Ker}(B(z,\lambda))$, then, by definition, $\psi = \lambda W^{1/2}\phi$ and

$$0 = (T - z)\phi + W^{1/2}\psi = (T - z + \lambda W)\phi.$$

Therefore, $\phi \in \operatorname{Ker}(T_\lambda - z)$ and $\Phi(\phi) = \phi \oplus \psi$. Since Φ is clearly injective, it follows that

$$\Phi : \operatorname{Ker}(T_\lambda - z) \to \operatorname{Ker}(B(z,\lambda))$$

is an isomorphism.

(ii) Note that

$$P^\perp B(z,\lambda)P^\perp(\phi \oplus \psi) = P^\perp B(z,\lambda)\phi \oplus 0 = P^\perp\big((T - z)\phi \oplus W^{1/2}\phi\big) = (T - z)\phi \oplus 0.$$

Since $z \notin \sigma(T)$, $P^\perp B(z,\lambda)P^\perp = (T - z) \oplus 0$, restricted to $\operatorname{Ran}(P^\perp) = \mathcal{D} \oplus 0$, is boundedly invertible, and

$$(P^\perp B(z,\lambda)P^\perp)^{-1}P^\perp = (T - z)^{-1} \oplus 0.$$

Thus, the Feshbach–Schur map exists and is given by

$$\begin{aligned} F_P(B(z,\lambda)) &= PB(z,\lambda)P - PB(z,\lambda)P^\perp(P^\perp B(z,\lambda)P^\perp)^{-1}P^\perp B(z,\lambda)P \\ &= 0 \oplus (-\lambda^{-1}) - 0 \oplus W^{1/2}(T - z)^{-1}W^{1/2} \\ &= 0 \oplus -\big(\lambda^{-1} + W^{1/2}(T - z)^{-1}W^{1/2}\big). \end{aligned} \tag{14}$$

Hence,

$$\operatorname{Ker}(B(z,\lambda)) \quad \text{and} \quad \operatorname{Ker}(F_P(B(z,\lambda))) = \operatorname{Ker}\big(\lambda^{-1} + W^{1/2}(T - z)^{-1}W^{1/2}\big)$$

are isomorphic.

(iii) By items (i) and (ii), one has that $\operatorname{Ker}(T_\lambda - z)$ and

$$\operatorname{Ker}\big(\lambda^{-1} + W^{1/2}(T - z)^{-1}W^{1/2}\big)$$

are isomorphic. Therefore, $z \in \sigma_p(T_\lambda)$ if and only if $\operatorname{Ker}(T_\lambda - z)$ is nontrivial, if and only if

$$\operatorname{Ker}\big(\lambda^{-1} + W^{1/2}(T - z)^{-1}W^{1/2}\big)$$

is nontrivial, if and only if

$$-1 \in \sigma_p\big(\lambda W^{1/2}(T - z)^{-1}W^{1/2}\big) = \sigma\big(\lambda W^{1/2}(T - z)^{-1}W^{1/2}\big).$$

Exercise 7.3 (6)

Let $\Omega \subset \mathbb{C}$ be an open, connected set and $\mathcal{H}$ be a Hilbert space. Consider

$$T : \Omega \subset \mathbb{C} \to \mathcal{B}(H),$$

an analytic function such that $T(z)$ is a compact operator for all $z \in \Omega$ and that there exists $z_0 \in \Omega$ such that $1 \notin \sigma(T(z_0))$. Consider the set

$$N_T := \{z \in \Omega : 1 \in \sigma(T(z))\}. \tag{15}$$

(i) Show that $\Omega \setminus N_T \subset \mathbb{C}$ is an open set and that the map

$$(1-T)^{-1} : \Omega \setminus N_T \to \mathcal{B}(H), \tag{16}$$

is an analytic function.

(ii) For $\zeta \in \Omega$ write $T(\zeta) = F_\zeta + Q_\zeta$ with F_ζ finite rank operator and $\|Q_\zeta\|_{op} < 1$. Show that there exists $r > 0$ such that for all $z \in D(\zeta, r) \subset \Omega$, $\|Q(z)\|_{op} \leq 1$ where $Q : D(\zeta, r) \to \mathcal{B}(\mathcal{H})$ is given by

$$Q(z) = T(z) - F_\zeta. \tag{17}$$

Conclude that $1 - T(z)$ is invertible if and only if $1 - F_\zeta(1 - Q(z))^{-1}$ is invertible.

(iii) Let P be the orthogonal projection onto $\mathrm{Ran}(F_\zeta)$. Show that the Feshbach-Schur map $F_P(1 - F_\zeta(1 - Q(z))^{-1})$ exists. Conclude that N_T is a discrete set.

(iv) Prove that for all $\zeta \in N_T$ there exists $r > 0$, $n_0 \in \mathbb{Z}$ and $\{P_n\}_{n=n_0}^{\infty}$ bounded operators with P_n finite rank operator for $n < 0$, such that for all $z \in D(\zeta, r) \setminus \{\zeta\}$ one has

$$(1 - T(z))^{-1} = \sum_{n=n_0}^{\infty} (z - \zeta)^n P_n. \tag{18}$$

Solution.

(i) Consider $z \in \Omega \setminus N_T$. Then, by definition, $1 - T(z)$ is invertible. Let $\zeta \in \Omega$. Then one has

$$\begin{aligned} 1 - T(\zeta) &= 1 - T(z) - (T(\zeta) - T(z)) \\ &= \left[1 - (T(\zeta) - T(z))(1 - T(z))^{-1}\right](1 - T(z)). \end{aligned} \tag{19}$$

Since T is analytic, and in particular continuous, there exists $r > 0$ such that

$$\|T(\zeta) - T(z)\|_{op} \leq \frac{1}{2}\|(1 - T(z))^{-1}\|_{op}^{-1}, \qquad \zeta \in D(z; r). \tag{20}$$

Then Eqs. (19), (20), and the Neumann series imply that, for all $\zeta \in D(z, r)$, the operator $1 - T(\zeta)$ is invertible. Hence $D(z, r) \subset \Omega \setminus N_T$, and therefore $\Omega \setminus N_T$ is open. One easily checks that

$$1 - T : \Omega \setminus N_T \to \mathcal{B}(H)$$

is analytic and that

$$\frac{d}{dz}(1 - T(z))^{-1} = (1 - T(z))^{-1} T'(z) (1 - T(z))^{-1}.$$

(ii) Let $\zeta \in \Omega$. Since $T(\zeta)$ is compact, there exists a finite-rank operator F_ζ and an operator $Q_\zeta \in \mathcal{B}(H)$ with $\|Q_\zeta\|_{op} < 1$ such that

$$T(\zeta) = F_\zeta + Q_\zeta.$$

For all $z \in \Omega$, one then has

$$Q(z) := T(z) - F_\zeta = T(z) - T(\zeta) + T(\zeta) - F_\zeta = T(z) - T(\zeta) + Q_\zeta. \tag{21}$$

Since T is continuous, there exists $r > 0$ such that, for all $z \in D(\zeta; r) \subset \Omega$,

$$\|T(z) - T(\zeta)\| < 1 - \|Q_\zeta\|_{op}.$$

Thus, by Eq. (21), one obtains

$$\|Q(z)\| < 1, \qquad z \in D(\zeta; r).$$

In particular, for all $z \in D(\zeta; r)$, the operator $1 - Q(z)$ is invertible, and one has

$$1 - T(z) = 1 - Q(z) - F_\zeta = \big(1 - F_\zeta(1 - Q(z))^{-1}\big)(1 - Q(z)). \tag{22}$$

Therefore, $1 - T(z)$ is invertible if and only if $1 - F_\zeta(1 - Q(z))^{-1}$ is invertible.

(iii) Let P be the orthogonal projection onto $\mathrm{Ran}(F_\zeta)$, and let $P^\perp = 1 - P$. Then

$$P^\perp\big(1 - F_\zeta(1 - Q(z))^{-1}\big)P^\perp = P^\perp,$$

which, restricted to $\mathrm{Ran}(P^\perp)$, is invertible with inverse $I_{\mathrm{Ran}(P^\perp)}$. Thus, the Feshbach–Schur map

$$F_P\big(1 - F_\zeta(1 - Q(z))^{-1}\big)$$

exists and is given by

$$F_P\big(1 - F_\zeta(1 - Q(z))^{-1}\big) = P - F_\zeta(1 - Q(z))^{-1}P. \tag{23}$$

Using item (ii), one obtains that $1 - T(z)$ is invertible if and only if $1 - F_\zeta(1 - Q(z))^{-1}$ is invertible, if and only if

$$F_P\big(1 - F_\zeta(1 - Q(z))^{-1}\big)$$

is invertible. In other words, for $z \in D(\zeta; r)$ one has

$$1 \in \sigma(T(z)) \quad \text{if and only if} \quad p(z) := \det\big(P - F_\zeta(1 - Q(z))^{-1}P\big) = 0. \tag{24}$$

Since Ω is connected, there exist finitely many points

$$z_0 = \zeta_0, \zeta_1, \dots, \zeta_n = \zeta$$

in Ω, and radius $r_0, r_1, \dots, r_n$, such that

$$D(\zeta_j; r_j) \cap D(\zeta_{j+1}; r_{j+1}) \neq \emptyset$$

and, for all $z \in D(\zeta_j; r_j) \subset \Omega$,

$$1 \in \sigma(T(z)) \quad \text{if and only if} \quad p_j(z) := \det\big(P_j - F_{\zeta_j}(1 - Q_{\zeta_j}(z))^{-1}P_j\big) = 0. \tag{25}$$

If N_T has an accumulation point at $\zeta \in \Omega$, then Eq. (25) implies that the set of zeros of the analytic function

$$p_n : D(\zeta; r_n) \to \mathbb{C}$$

has an accumulation point. Hence $p_n(z) = 0$ for all $z \in D(\zeta; r_n)$. Then, Eq. (25) implies

$$1 \in \sigma(T(z)), \quad z \in D(\zeta; r_n) \cap D(\zeta_{n-1}; r_{n-1}) \neq \emptyset,$$

which implies (together with Eq. (25)) that the set of zeros of p_{n-1} has an acumulation point then

$$p_{n-1}(z) = 0, \qquad z \in D(\zeta_{n-1}; r_{n-1}).$$

Repeating the above argument, inductively, one obtains

$$p_0(z) = 0, \qquad z \in D(z_0; r_0).$$

Therefore $1 \in \sigma(T(z_0))$, which contradicts the hypothesis. This implies that N_T is discrete.

(iv) For $\zeta \in N_T$, take $r > 0$ such that

$$D(\zeta, r) \setminus \{\zeta\} \subset \Omega \setminus N_T$$

and such that Eq. (24) is satisfied. Then, for all $z \in D(\zeta, r) \setminus \{\zeta\}$, the operator $1 - F_\zeta(1 - Q(z))^{-1}$ is invertible, and

$$\left(1 - F_\zeta(1 - Q(z))^{-1}\right)^{-1} = \left(P - F_\zeta(1 - Q(z))^{-1}P\right)^{-1}\left(P + F_\zeta(1 - Q(z))^{-1}P^\perp\right) + P^\perp. \tag{26}$$

Since

$$\left(P - F_\zeta(1 - Q(z))^{-1}P\right)^{-1} : \mathrm{Ran}(F_\zeta) \to \mathrm{Ran}(F_\zeta)$$

is a meromorphic function, with a pole at ζ, one has

$$\left(P - F_\zeta(1 - Q(z))^{-1}P\right)^{-1} = \sum_{n=n_0}^{\infty} (z - \zeta)^n R_n \tag{27}$$

for some $n_0 \in \mathbb{Z}$ and some operators

$$R_n : \mathrm{Ran}(F_\zeta) \to \mathrm{Ran}(F_\zeta).$$

Moreover, all the other functions appearing in Eq. (26) are analytic on $D(\zeta; r)$. Also, $(1 - Q(z))^{-1}$ is analytic on $D(\zeta; r)$. Therefore, Eqs. (22), (26), and (27) imply the result.