Technische Universität Braunschweig

6. Exercise Sheet 6

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Exercise 6.1 (6)

Let $A, W \in \mathfrak{M}_{3\times 3}(\mathbb{C})$ and define $A_g := A + gW$ with $g \in \mathbb{R}$ as in Eqs. (V.47)–(V.49) of the lecture notes, and let $z \in D(0,1)$ be given by (V.60), where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} 0 & i & 2 \\ -i & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}. \tag{1}$$

Determine $\rho := \|W\|_{op}^{-1} < \infty$ and compute $a_0, a_1, a_2, a_3 \in \mathbb{C}$ and $C_4 < \infty$ such that

$$\left| z - (a_0 + a_1 g + a_2 g^2 + a_3 g^3) \right| \leq C_4 g^4$$

for all $g \in (-\rho, \rho)$.

Exercise 6.2 (6)

Let $n \in \mathbb{N}$, $T = T^*, W = W^* \in \mathcal{B}(\mathbb{C}^n) \setminus \{0\}$ and $T_g := T + gW$ with $g \in \mathbb{R}$. Let $T_{k,k} = \langle e_k \mid Te_k \rangle = 0$ and $T_{k,\ell} = \langle e_k \mid Te_\ell \rangle < 0$ for all $k \neq l \in \{1, \ldots, n\}$, where $\{e_k\}_{k=1}^n \subseteq \mathbb{C}^n$ denotes the canonical orthonormal basis.

(i) Let $\inf \sigma(T) =: \lambda_0$. Show that λ_0 is a strictly negative eigenvalue.

(ii) Let $\psi \in \mathrm{Ker}[T - \lambda_0] \setminus \{0\}$ be an eigenvector corresponding to λ_0. Show that there exists $\alpha \in [0, 2\pi)$ such that $e^{-i\alpha}\langle e_k \mid \psi \rangle > 0$ for all $k \in \{1, \ldots, n\}$. Conclude that λ_0 is a simple eigenvalue of T.

(iii) Assume that $\langle \psi \mid W\psi \rangle = 0$, $W\psi \neq 0$ and $\inf \sigma(T_g) =: \lambda_g$. Show that the map $\mathbb{R} \ni g \mapsto \lambda_g \in \mathbb{R}$ attains a strict local maximum at $g = 0$.

Exercise 6.3 (12)

Let $n \in \mathbb{N}$, $T = T^*, W = W^* \in \mathcal{B}(\mathbb{C}^n) \setminus \{0\}$, and $T_g := T + gW$, with $g \geq 0$ and $W \geq 0$. Let $\lambda_0 := \inf \sigma(T)$ be a simple eigenvalue with normalized eigenvector $\psi_0 \in \mathbb{C}^n$ and associated orthogonal projection $P_0 := |\psi_0\rangle\langle\psi_0|$. Furthermore, define $R_0^\perp(z) := (T_0 - z)^{-1} P_0^\perp$ and $\lambda_g := \inf \sigma(T_g)$.

(i) Show that there exists $g_0 > 0$ such that λ_g is a simple eigenvalue for $g \in (-g_0, g_0)$.

(ii) Show that there exists $g_0 > 0$ such that λ_g satisfies the estimate

$$\lambda_g \geq \lambda_0 + g\langle\psi_0|W\psi_0\rangle - g^2 \left\langle \psi_0 \middle| W\, R_0^\perp \left[\lambda_0 + g\langle\psi_0|W\psi_0\rangle\right] W\psi_0 \right\rangle \tag{2}$$

for $g \in [0, g_0)$.

(iii) Show that there exists $g_0 > 0$ such that λ_g satisfies the estimate

$$\lambda_g \leq \lambda_0 + g\langle\psi_0|W\psi_0\rangle - g^2 \left\langle \psi_0 \middle| W R_0^\perp(\lambda_0) W\psi_0 \right\rangle + g^3 \left\langle \psi_0 \middle| W R_0^\perp(\lambda_0) W R_0^\perp(\lambda_0) W\psi_0 \right\rangle \tag{3}$$

for $g \in [0, g_0)$. Hint: Apply the variational principle with the trial vector $\psi_0 - gR_0^\perp(\lambda_0)W\psi_0$. (Why is this a good choice?)