



5. Exercise Sheet 5

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Let $d \in \{1, 2, 3\}$ and $V \in L^2(\mathbb{R}^d; \mathbb{R}_0^+)$ vanish at infinity, i.e., with $V_R(x) := \mathbb{1}(|x| \geq R) V(x)$, it holds that $\lim_{R \rightarrow \infty} \|V_R\|_\infty = 0$.

The goal of this exercise is to show that the eigenfunctions of $H = -\Delta - V(x)$ on $\mathfrak{H} := L^2(\mathbb{R}^d)$ corresponding to eigenvalues below $-E < 0$ decay exponentially at infinity like $e^{-\tau\sqrt{E}|x|}$ (for a universal $\tau > 0$ chosen as large as possible).

Exercise 5.1 (6)

Let $\phi \in C^\infty(\mathbb{R}^+; \mathbb{R}^+)$ be a convex function with $\phi(r) = 0$ for $r \leq 2$ and $\phi(r) = r - 3$ for $r \geq 4$. Furthermore, let $\alpha > 0$, $R > 1$, and let $F : \mathbb{R}^d \rightarrow \mathbb{R}^+$ be given by $F(x) := \alpha R \phi(|x|/R)$.

- (i) Compute ∇F and ΔF , and show that $\|\nabla F\|_\infty \leq \alpha$ and $\|\Delta F\|_\infty \leq C\alpha/R$ hold, where the constant $C < \infty$ depends only on d and $\|\phi''\|_\infty$.
- (ii) Let $(\Delta_F, C_0^\infty(\mathbb{R}^d)) \in \mathfrak{L}[\mathfrak{H}]$ be defined by $\Delta_F := e^F \circ \Delta \circ e^{-F}$. Show that $(W_F, C_0^\infty(\mathbb{R}^d))$ with $W_F := \Delta_F - \Delta$ is an infinitesimal perturbation of $(\Delta, C_0^\infty(\mathbb{R}^d)) \in \mathfrak{L}[\mathfrak{H}]$.
- (iii) Conclude that $(\Delta_F, C_0^\infty(\mathbb{R}^d)) \in \mathfrak{L}[\mathfrak{H}]$ can be extended to a closed operator $(\Delta_F, H^2(\mathbb{R}^d)) \in \mathfrak{L}[\mathfrak{H}]$.

Solution.

- (i) An explicit computation shows that

$$(\nabla F)(x) = \alpha \phi'(|x|/R) \frac{x}{|x|}, \quad (1)$$

and

$$(\Delta F)(x) = \alpha \frac{1}{R} \phi''(|x|/R) + \alpha \frac{d-1}{|x|} \phi'(|x|/R). \quad (2)$$

Since ϕ is convex, one has $0 \leq \phi'(r) \leq \phi'(4) = 1$, $0 \leq r \leq 4$. Hence $\|\phi'\|_\infty = 1$. Using Eq. (1), it follows that $\|\nabla F\|_\infty \leq \alpha$. Moreover, since $\phi'(r) = 0$ for $r \leq 2$, one has

$$\frac{1}{|x|} \phi'(|x|/R) \leq \frac{1}{2R} \phi'(|x|/R). \quad (3)$$

Combining Eqs. (2) and (3), we obtain

$$|(\Delta F)(x)| \leq \frac{\alpha}{R} \left(\|\phi''\|_\infty + \frac{d-1}{2} \right) = C\alpha/R. \quad (4)$$

- (ii) Let $g \in C_0^\infty(\mathbb{R}^d)$. An explicit computation shows that

$$\Delta_F g = -g\Delta F + |\nabla F|^2 g - 2\nabla g \cdot \nabla F + \Delta g. \quad (5)$$

Therefore,

$$\|W_F g\|_{L^2} = \|(\Delta_F - \Delta)g\|_{L^2} \leq (\|\Delta F\|_\infty + \|\nabla F\|_\infty^2) \|g\|_{L^2} + 2\|\nabla F\|_\infty \|\nabla g\|_{L^2}. \quad (6)$$

Furthermore, for every $c > 0$,

$$\begin{aligned} \|\nabla g\|_{L^2}^2 &= \langle g, -\Delta g \rangle_{L^2} \leq \|g\|_{L^2} \|\Delta g\|_{L^2} \\ &\leq \frac{c^2}{2} \|\Delta g\|_{L^2}^2 + \frac{1}{2c^2} \|g\|_{L^2}^2. \end{aligned} \quad (7)$$

Using Eqs. (6) and (7), it follows that $(W_F, C_0^\infty(\mathbb{R}^d))$ is an infinitesimal perturbation of $(\Delta, C_0^\infty(\mathbb{R}^d))$.

(iii) Let $\phi \in H^2(\mathbb{R}^d)$. Then there exists a sequence $(g_n)_{n \in \mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ such that

$$\|g_n - \phi\|_{L^2} \rightarrow 0, \quad \|\Delta g_n - \Delta \phi\|_{L^2} \rightarrow 0.$$

Eqs. (6) and (7) imply that $(W_F g_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. We define

$$W_F \phi := \lim_{n \rightarrow \infty} W_F g_n,$$

and

$$\Delta_F \phi := W_F \phi + \Delta \phi, \quad \phi \in H^2(\mathbb{R}^d). \quad (8)$$

Note that Eq. (7) extends to $g \in H^2(\mathbb{R}^d)$ by approximating $D^\alpha g$, $|\alpha| \leq 2$, with functions $D^\alpha g_n \in C_0^\infty(\mathbb{R}^d)$ and passing to the limit. Consequently, Eq. (6) also extends to $H^2(\mathbb{R}^d)$. Therefore, $(W_F, H^2(\mathbb{R}^d))$ is an infinitesimal perturbation of $(\Delta, H^2(\mathbb{R}^d))$.

We now show that $(\Delta_F, H^2(\mathbb{R}^d))$ is closed. Let $(\phi_n)_{n \in \mathbb{N}}$ be a sequence in $H^2(\mathbb{R}^d)$ such that

$$\phi_n \rightarrow \phi \in L^2, \quad \Delta_F \phi_n \rightarrow \psi \in L^2.$$

Since $(W_F, H^2(\mathbb{R}^d))$ is an infinitesimal perturbation of $(\Delta, H^2(\mathbb{R}^d))$, for every $a < 1$ there exists $b > 0$ such that

$$\|\Delta f\| - \|\Delta_F f\| \leq \|W_F f\| \leq a \|\Delta f\| + b \|f\|, \quad f \in H^2(\mathbb{R}^d). \quad (9)$$

In particular,

$$\|\Delta \phi_n\| \leq (1-a)^{-1} \|\Delta_F \phi_n\| + (1-a)^{-1} b \|\phi_n\|.$$

Hence $(\Delta \phi_n)_{n \in \mathbb{N}}$ is a Cauchy sequence and therefore convergent. Since $\phi_n \rightarrow \phi$ and $(\Delta, H^2(\mathbb{R}^d))$ is closed, it follows that $\phi \in H^2(\mathbb{R}^d)$ and

$$\Delta \phi_n \rightarrow \Delta \phi.$$

Consequently,

$$W_F \phi_n \rightarrow W_F \phi,$$

and Eq. (8) yields

$$\Delta_F \phi_n \rightarrow W_F \phi + \Delta \phi = \Delta_F \phi.$$

Therefore, $\psi = \Delta_F \phi$, which proves that $(\Delta_F, H^2(\mathbb{R}^d))$ is closed.

Exercise 5.2 (6)

Let $\alpha > 0$, $R > 1$, and $z = -a + ib \in \mathbb{C}$ with $a > 0$ and $b \in \mathbb{R}$. Show that $-\Delta_F - z$ is boundedly invertible for fixed $a > 0$ and sufficiently small $\alpha > 0$, and provide a norm bound.

Solution. Note that $-\Delta_F - z = -\Delta - W_F - z$. For all $a > 0$, the operator $-\Delta - z$ is boundedly invertible, hence

$$-\Delta_F - z = -\Delta - W_F - z = (1 - W_F(-\Delta - z)^{-1})(-\Delta - z). \quad (10)$$

For all $\phi \in L^2(\mathbb{R}^d)$, using Eq. (6) and (7), one obtains for $\phi \in L^2(\mathbb{R}^d)$ and $c > 0$,

$$\begin{aligned} \|W_F(-\Delta - z)^{-1}\phi\|_{L^2} &\leq (C\alpha/R + \alpha^2) \|(-\Delta - z)^{-1}\phi\|_{L^2} \\ &\quad + \alpha \sqrt{\|(-\Delta - z)^{-1}\phi\|_{L^2} \|-\Delta(-\Delta - z)^{-1}\phi\|_{L^2}} \\ &\leq (C\alpha/R + \alpha^2) a^{-1} \|\phi\|_{L^2} + 2\alpha a^{1/2} \|\phi\|_{L^2}, \end{aligned}$$

where we used that $\|(-\Delta - z)^{-1}\|_{op} \leq 1/a$ and $\|-\Delta(-\Delta - z)^{-1}\|_{op} \leq 1$.

Then one obtains

$$\|W_F(-\Delta - z)^{-1}\|_{op} \leq (C\alpha/R + \alpha^2)a^{-1} + 2\alpha a^{-1/2} = n_\alpha(a). \quad (11)$$

Taking $\alpha > 0$ such that $n_\alpha(a) < 1$, it follows that $1 - W_F(-\Delta - z)^{-1}$ is invertible. Then Eq. (10) implies that $-\Delta_F - z$ is boundedly invertible. Moreover, one has

$$\|(-\Delta_F - z)^{-1}\|_{op} \leq a^{-1} \sum_{k=0}^{\infty} n_\alpha(a)^k = a^{-1} \frac{1}{1 - n_\alpha(a)}. \quad (12)$$

Exercise 5.3 (12)

Let $E > 0$ and let $P := \mathbb{1}[H < -E]$ and $P_R := \mathbb{1}[H_R < -E]$ be the spectral projections of $H = H^*$ and $H_R = H_R^* := -\Delta - V_R(x)$, respectively, onto $(-\infty, -E]$.

- (i) Show that $\dim \text{Ran}(P_R) \leq \dim \text{Ran}(P) < \infty$ holds.
- (ii) Show that $P_R = 0$ holds if $R < \infty$ is sufficiently large.
- (iii) Show that there exist constants $C_E < \infty$ and $\tau > 0$ such that $\|\exp(\tau\sqrt{E}|x|)P\|_{op} \leq C_E$.

Hint for (iii): Write P as a Dunford–Cauchy integral and use $P = P - P_R$ for $R < \infty$ sufficiently large. Now, choose α appropriately and estimate the norm of $e^F P = e^F (P - P_R)$ using Exercise 5.2 and $e^{-F}(V - V_R) = (V - V_R)$.

Solution.

- (i) The Kato–Rellich theorem implies that H_R and H are bounded below self-adjoint operators. Since V_R and V are relatively compact with respect to $-\Delta$, one has

$$\sigma_{ess}(H) = \sigma_{ess}(H_R) = \sigma(-\Delta) = [0, \infty). \quad (13)$$

Since $V_R \leq V$, it follows that $H \leq H_R$. Hence, the min-max principle implies that for all $E > 0$,

$$\dim(\text{Ran}(P_R)) \leq \dim(\text{Ran}(P)).$$

Moreover, since H is bounded below, one has $\sigma(H) \subset [M, \infty)$ for some $M < 0$. Since $\sigma(H)$ is closed, for all $E > 0$ the set

$$\sigma(H) \cap (-\infty, -E] = \sigma(H) \cap [M, -E] = \sigma_{dis}(H) \cap [M, -E]$$

is finite. Therefore, $\dim \text{Ran}(P) < \infty$.

(ii) One has $H_R = -\Delta - V_R \geq -\|V_R\|_\infty$, which implies that

$$\sigma(H_R) \subset [-\|V_R\|_\infty, \infty).$$

Let R be sufficiently large such that $\|V_R\|_\infty < E$. Then

$$(-\infty, -E) \cap \sigma(H_R) \subset (-\infty, -E) \cap [-\|V_R\|_\infty, \infty) = \emptyset.$$

Hence $P_R = 0$.

(iii) Let

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < -E$$

be the eigenvalues of H lying below $-E$. Let $r > 0$ be such that $\lambda_n + r < -E$. Let

$$Q := [\lambda_1 - r, \lambda_n + r] \times [-1, 1],$$

and let $\Gamma := \partial Q$ denote its boundary, oriented counterclockwise. Then

$$P := -\frac{1}{2\pi i} \int_\Gamma (H - z)^{-1} dz. \quad (14)$$

Let $R > 0$ be sufficiently large such that $P_R = 0$. Hence

$$\begin{aligned} P &= (P - P_R) = -\frac{1}{2\pi i} \int_\Gamma ((H - z)^{-1} - (H_R - z)^{-1}) dz \\ &= -\frac{1}{2\pi i} \int_\Gamma (H_R - z)^{-1} (V - V_R) (H - z)^{-1} dz \end{aligned} \quad (15)$$

Since $V \in L^2(\mathbb{R}^d)$, it follows that V is an infinitesimal perturbation of $-\Delta$. Consequently, V is also an infinitesimal perturbation of $H = -\Delta - V$. Together with the fact that $\|V_R\|_\infty < \infty$, this implies that

$$\sup_{z \in \Gamma} \|(V - V_R)(H - z)^{-1}\|_{op} < C_1 < \infty. \quad (16)$$

Let $\alpha > 0$ be given by $\alpha = (\sqrt{2} - 1)E^{1/2}$. Then one obtains

$$\sup \left\{ \frac{\alpha^2}{-\Re e(z)} + 2 \frac{\alpha}{(-\Re e(z))^{1/2}} : \Re e(z) \leq \lambda_n + r \right\} =: \tilde{C} < \frac{\alpha^2}{E} + \frac{2\alpha}{E^{1/2}} = 1. \quad (17)$$

Take $R > 0$ sufficiently large such that

$$n_\alpha(E) := C\alpha(ER)^{-1} + \tilde{C} < 1,$$

where $C > 0$ is given by Eq. (4), and such that

$$\hat{C} := \|V_R\|_\infty \frac{E^{-1}}{1 - n_\alpha(E)} < 1. \quad (18)$$

Consider $F(x) = \alpha R \phi(|x|/R)$ as in Exercise 5.1. Using Exercise 5.2, one obtains that for all $z \in \mathbb{C}$ with $\Re e(z) \leq \lambda_n + r$, the operator $-\Delta_F - z$ is boundedly invertible and

$$\|(-\Delta_F - z)^{-1}\|_{op} \leq E^{-1} \frac{1}{1 - n_\alpha(E)}. \quad (19)$$

Since

$$e^F H_R e^{-F} - z = -\Delta_F - z - V_R = (1 - V_R(-\Delta_F - z)^{-1})(-\Delta_F - z), \quad (20)$$

it follows that if $\Re(z) < \lambda_n + r$, then $e^F H_R e^{-F} - z$ is boundedly invertible and

$$\sup_{\Re(z) \leq \lambda_n + r} \|(e^F H_R e^{-F} - z)^{-1}\|_{op} \leq E^{-1} \frac{1}{1 - n_\alpha(E)} \frac{1}{1 - \hat{C}} =: C_2 < \infty. \quad (21)$$

Note that $e^{-F}(V - V_R) = V - V_R$, then one obtains

$$\begin{aligned} e^F (H_R - z)^{-1} (V - V_R) (H - z)^{-1} &= e^F (H_R - z)^{-1} e^{-F} (V - V_R) (H - z)^{-1} \\ &= (e^F H_R e^{-F} - z)^{-1} (V - V_R) (H - z)^{-1} \end{aligned} \quad (22)$$

Using Eqs. (15) and (22) one obtains

$$e^F P = -\frac{1}{2\pi i} \int_{\Gamma} (e^F H_R e^{-F} - z)^{-1} (V - V_R) (H - z)^{-1} dz. \quad (23)$$

Using Eqs. (16) and (21) one obtains

$$\|e^F P\|_{op} \leq \frac{|\Gamma|}{2\pi} C_1 C_2 < \infty. \quad (24)$$

Finally, using that $e^{\alpha|x| - 3R} \leq e^{F(x)}$, one obtains

$$\|e^{\alpha|x|} P\|_{op} = \|e^{\alpha|x| - F(x)} e^F P\|_{op} \leq e^{3R} \frac{|\Gamma|}{2\pi} C_1 C_2. \quad (25)$$