Technische Universität Braunschweig

5. Exercise Sheet 5

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Let $d \in \{1,2,3\}$ and $V \in L^2(\mathbb{R}^d;\mathbb{R}_0^+)$ vanish at infinity, i.e., with $V_R(x) := \mathbb{1}\big(|x| \geq R\big)\, V(x)$, it holds that $\lim_{R\to\infty} \|V_R\|_\infty = 0$.

The goal of this exercise is to show that the eigenfunctions of $H = -\Delta - V(x)$ on $\mathfrak{H} := L^2(\mathbb{R}^d)$ corresponding to eigenvalues below $-E < 0$ decay exponentially at infinity like $e^{-\tau\sqrt{E}|x|}$ (for a universal $\tau > 0$ chosen as large as possible).

Exercise 5.1 (6)

Let $\phi \in C^\infty(\mathbb{R}^+;\mathbb{R}^+)$ be a convex function with $\phi(r) = 0$ for $r \leq 2$ and $\phi(r) = r - 3$ for $r \geq 4$. Furthermore, let $\alpha > 0$, $R > 1$, and let $F : \mathbb{R}^d \to \mathbb{R}^+$ be given by $F(x) := \alpha R\, \phi\big(|x|/R\big)$.

(i) Compute ∇F and ΔF, and show that $\|\nabla F\|_\infty \leq \alpha$ and $\|\Delta F\|_\infty \leq C\alpha/R$ hold, where the constant $C < \infty$ depends only on d and $\|\phi''\|_\infty$.

(ii) Let $\big(\Delta_F, C_0^\infty(\mathbb{R}^d)\big) \in \mathfrak{L}[\mathfrak{H}]$ be defined by $\Delta_F := e^F \circ \Delta \circ e^{-F}$. Show that $\big(W_F, C_0^\infty(\mathbb{R}^d)\big)$ with $W_F := \Delta_F - \Delta$ is an infinitesimal perturbation of $\big(\Delta, C_0^\infty(\mathbb{R}^d)\big) \in \mathfrak{L}[\mathfrak{H}]$.

(iii) Conclude that $\big(\Delta_F, C_0^\infty(\mathbb{R}^d)\big) \in \mathfrak{L}[\mathfrak{H}]$ can be extended to a closed operator $\big(\Delta_F, H^2(\mathbb{R}^d)\big) \in \mathfrak{L}[\mathfrak{H}]$.

Exercise 5.2 (6)

Let $\alpha > 0$, $R > 1$, and $z = -a + ib \in \mathbb{C}$ with $a > 0$ and $b \in \mathbb{R}$. Show that $-\Delta_F - z$ is boundedly invertible for fixed $a > 0$ and sufficiently small $\alpha > 0$, and provide a norm bound.

Exercise 5.3 (12)

Let $E > 0$ and let $P := \mathbb{1}[H < -E]$ and $P_R := \mathbb{1}[H_R < -E]$ be the spectral projections of $H = H^*$ and $H_R = H_R^* := -\Delta - V_R(x)$, respectively, onto $(-\infty, -E]$.

(i) Show that $\dim \mathrm{Ran}(P_R) \leq \dim \mathrm{Ran}(P) < \infty$ holds.

(ii) Show that $P_R = 0$ holds if $R < \infty$ is sufficiently large.

(iii) Show that there exist constants $C_E < \infty$ and $\tau > 0$ such that $\big\| \exp\big(\tau\sqrt{E}|x|\big)\, P\big\|_{op} \leq C_E$.

Hint for (iii): Write P as a Dunford-Cauchy integral and use $P = P - P_R$ for $R < \infty$ sufficiently large. Now, choose α appropriately and estimate the norm of $e^F\, P = e^F\, (P - P_R)$ using Exercise 5.2 and $e^{-F}(V - V_R) = (V - V_R)$.