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4. Exercise Sheet 4

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Exercise 4.1 (6)

Let

$$B := \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{C}).$$

Compute the diagonal form of B, if B is diagonalizable, or otherwise compute the Jordan normal form of B and the eigenprojections of B.

Solution. The eigenvalues of B are $\{1,2\}$. Hence, B is diagonalizable, and its diagonal form is

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

For every $z \notin \{1,2\}$, the resolvent of B is given by

$$(B-z)^{-1} = (1-z)^{-1}(2-z)^{-1}\begin{pmatrix} 2-z & -3 \\ 0 & 1-z \end{pmatrix}. \tag{1}$$

Let $\gamma_1, \gamma_2 : [0,2\pi] \to \mathbb{C}$ be the positively oriented curves defined by

$$\gamma_1(t) = 1 + \frac{1}{2}e^{it}, \qquad \gamma_2(t) = 2 + \frac{1}{2}e^{it}.$$

Then,

$$-\frac{1}{2\pi i}\int_{\gamma_1}(B-z)^{-1}\,dz = \begin{pmatrix} \frac{1}{2\pi i}\int_{\gamma_1}(z-1)^{-1}\,dz & 3\,\frac{1}{2\pi i}\int_{\gamma_1}(1-z)^{-1}(2-z)^{-1}\,dz \\ 0 & \frac{1}{2\pi i}\int_{\gamma_1}(z-2)^{-1}\,dz \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}.$$

Similarly,

$$-\frac{1}{2\pi i}\int_{\gamma_2}(B-z)^{-1}\,dz = \begin{pmatrix} \frac{1}{2\pi i}\int_{\gamma_2}(z-1)^{-1}\,dz & 3\,\frac{1}{2\pi i}\int_{\gamma_2}(1-z)^{-1}(2-z)^{-1}\,dz \\ 0 & \frac{1}{2\pi i}\int_{\gamma_2}(z-2)^{-1}\,dz \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix}.$$

Therefore, the eigenprojections of B are

$$\begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix}.$$

Exercise 4.2 (6)

Let $\Omega \subseteq \mathbb{C}$ be nonempty and open, and let $\mathcal{H}$ be a complex separable Hilbert space. A map

$$A : \Omega \to \mathcal{B}(\mathcal{H})$$

is called weakly analytic if, for every trace-class operator $\rho \in \mathcal{L}^1(\mathcal{H})$, the function

$$f_\rho : \Omega \to \mathbb{C}, \qquad f_\rho(z) := \operatorname{Tr}\{\rho A(z)\},$$

is analytic (in the sense of complex analysis).

Show that $A : \Omega \to \mathcal{B}(\mathcal{H})$ is analytic if and only if $A : \Omega \to \mathcal{B}(\mathcal{H})$ is weakly analytic.

Solution. Let us recall that a trace-class operator $\rho \in \mathcal{L}^1(\mathcal{H})$ is a bounded operator such that, for every orthonormal basis $\{\phi_n\}$ of $\mathcal{H}$, one has

$$\sum_{n=1}^{\infty} |\langle \rho\phi_n, \phi_n\rangle| < \infty. \tag{2}$$

For $\rho \in \mathcal{L}^1(\mathcal{H})$, one defines

$$\operatorname{Tr}(\rho) = \sum_{n=1}^{\infty} \langle \rho\phi_n, \phi_n\rangle, \tag{3}$$

which is independent of the choice of orthonormal basis. Moreover,

$$\|\rho\|_{\mathcal{L}^1} = \operatorname{Tr}(|\rho|).$$

Let us first prove the identity

$$\|A\|_{op} = \sup\left\{ |\operatorname{Tr}(\rho A)| : \|\rho\|_{\mathcal{L}^1(\mathcal{H})} = 1 \right\}, \qquad A \in \mathcal{B}(\mathcal{H}). \tag{4}$$

Let $\phi, \psi \in \mathcal{H}$ satisfy $\|\phi\| = \|\psi\| = 1$. Define the rank-one operator $\rho_{\phi,\psi}(h) = \langle \phi, h\rangle \psi$. Then $\rho_{\phi,\psi}$ is trace class and satisfies $\|\rho_{\phi,\psi}\|_{\mathcal{L}^1} = 1$. Moreover, $\operatorname{Tr}(\rho_{\phi,\psi} A) = \langle \phi, A\psi\rangle$. Hence,

$$|\langle \phi, A\psi\rangle| \le \sup\left\{ |\operatorname{Tr}(\rho A)| : \|\rho\|_{\mathcal{L}^1(\mathcal{H})} = 1 \right\}.$$

Therefore,

$$\|A\|_{op} = \sup\{|\langle \phi, A\psi\rangle| : \|\phi\| = \|\psi\| = 1\} \le \sup\left\{ |\operatorname{Tr}(\rho A)| : \|\rho\|_{\mathcal{L}^1(\mathcal{H})} = 1 \right\}. \tag{5}$$

On the other hand,

$$|\operatorname{Tr}(\rho A)| \le \operatorname{Tr}(|\rho A|) \le \|A\|_{op} \|\rho\|_{\mathcal{L}^1},$$

which implies

$$\sup\left\{ |\operatorname{Tr}(\rho A)| : \|\rho\|_{\mathcal{L}^1(\mathcal{H})} = 1 \right\} \le \|A\|_{op}. \tag{6}$$

$\Rightarrow$ Suppose that $A : \Omega \to \mathcal{B}(\mathcal{H})$ is analytic. Let $z_0 \in \Omega$. Since $\rho A'(z_0)$ is the limit of trace-class operators, it follows that

$$\rho A'(z_0) \in \mathcal{L}^1(\mathcal{H}).$$

Then,

$$
\begin{aligned}
&\lim_{h\to 0}\left[\frac{f_\rho(z_0+h)-f_\rho(z_0)}{h}-\operatorname{Tr}(\rho A'(z_0))\right]\\
&=\lim_{h\to 0}\operatorname{Tr}\left(\frac{1}{h}[\rho A(z_0+h)-\rho A(z_0)]-\rho A'(z_0)\right)\\
&=\operatorname{Tr}\left(\lim_{h\to 0}\left[\frac{1}{h}(\rho A(z_0+h)-\rho A(z_0))-\rho A'(z_0)\right]\right)\\
&=0.
\end{aligned}
\tag{7}
$$

Here we used the continuity of the map

$$\mathcal{L}^1(\mathcal{H})\ni\rho\mapsto\operatorname{Tr}(\rho).$$

Therefore, f_ρ is analytic and

$$f'_\rho(z_0)=\operatorname{Tr}(\rho A'(z_0)).$$

$\Leftarrow$ Let $z_0\in\Omega$, and let $\Gamma\subset\Omega$ be a closed curve surrounding z_0. For $\rho\in\mathcal{L}^1(\mathcal{H})$, since f_ρ is analytic, Cauchy's formula yields

$$\frac{f_\rho(z_0+h)-f_\rho(z_0)}{h}-f'_\rho(z_0)=\frac{1}{2\pi i}\int_\Gamma\left[\frac{1}{h}\left(\frac{1}{z-(z_0+h)}-\frac{1}{z-z_0}\right)-\frac{1}{(z-z_0)^2}\right]f_\rho(z)\,dz. \tag{8}$$

Using the estimate

$$|f_\rho(z)|=|\operatorname{Tr}(\rho A(z))|\le\|\rho\|_{\mathcal{L}^1}\|A(z)\|_{op},$$

Eq. (8) implies

$$\left|\operatorname{Tr}\left[\rho\left(\frac{A(z_0+h)-A(z_0)}{h}\right)\right]-f'_\rho(z_0)\right|\le\|\rho\|_{\mathcal{L}^1}\sup_{z\in\Gamma}\|A(z)\|_{op}\int_\Gamma\frac{|h|}{|z-z_0-h||z-z_0|^2}\,|dz|. \tag{9}$$

Therefore, for every $\rho\in\mathcal{L}^1(\mathcal{H})$ with $\|\rho\|_{\mathcal{L}^1}=1$, the family

$$\left\{\operatorname{Tr}\left[\rho\left(\frac{A(z_0+h)-A(z_0)}{h}\right)\right]\right\}_{h>0}$$

is Cauchy uniformly in ρ.

Hence, by Eq. (5), the family

$$\left\{\frac{A(z_0+h)-A(z_0)}{h}\right\}_{h>0}$$

is Cauchy in the operator norm. Consequently, it converges, and therefore A is analytic.

Exercise 4.3 (12)

Let $\Omega\subseteq\mathbb{C}$ be open and connected with $0\in\Omega$, let $\mathcal{H}$ be a complex separable Hilbert space, and let $\mathcal{D}\subseteq\mathcal{H}$ be a dense subspace. A mapping

$$(A,\mathcal{D}):\Omega\to\mathcal{L}[\mathcal{H}]$$

is called an analytic family of type A if $\mathcal{D}(\theta)=\mathcal{D}$ for all $\theta\in\Omega$ and

(a) For every $\theta\in\Omega$, the operator $(A(\theta),\mathcal{D})$ is closed and $\rho[A(\theta)]\neq\emptyset$.

(b) The map
$$A : \Omega \to \mathcal{B}(\mathcal{D};\mathcal{H})$$
is analytic, where the Hilbert space
$$(\mathcal{D}, \langle\cdot,\cdot\rangle_{\mathcal{D}})$$
is equipped with the graph inner product
$$\langle\varphi,\psi\rangle_{\mathcal{D}} := \langle A(0)\varphi, A(0)\psi\rangle + \langle\varphi,\psi\rangle.$$

Now let
$$\mathcal{H} = L^2(\mathbb{R}), \qquad \mathcal{D} := H^2(\mathbb{R}).$$
For $\theta \in \mathbb{R}$ and $\psi \in \mathcal{H}$, define
$$[U(\theta)\psi](x) := e^{-\theta/2}\psi(e^{-\theta}x)$$
and
$$[T(\theta)\psi](x) := \psi(x-\theta).$$

(i) Show that $U(\theta)$ and $T(\theta)$ are unitary for all $\theta \in \mathbb{R}$.

(ii) Compute
$$A(\theta) := U(\theta)\Big(-\frac{d^2}{dx^2}\Big)U(\theta)^*$$
and
$$B(\theta) := T(\theta)\Big(-\frac{d^2}{dx^2}\Big)T(\theta)^*$$
for every $\theta \in \mathbb{R}$.

(iii) Show that there exists $r > 0$ such that
$$A : \mathbb{R} \to \mathcal{B}(\mathcal{D};\mathcal{H})$$
admits an extension
$$A : S_r \to \mathcal{B}(\mathcal{D};\mathcal{H})$$
to an analytic family of type A on the strip
$$S_r := \mathbb{R} + i(-r,r),$$
and compute the spectrum $\sigma[A(\theta)]$ for every $\theta \in S_r$.

(iv) Show that there exists $r > 0$ such that
$$B : \mathbb{R} \to \mathcal{B}(\mathcal{D};\mathcal{H})$$
admits an extension
$$B : S_r \to \mathcal{B}(\mathcal{D};\mathcal{H})$$
to an analytic family of type A on the strip
$$S_r := \mathbb{R} + i(-r,r),$$
and compute the spectrum $\sigma[B(\theta)]$ for every $\theta \in S_r$.

Solution.

(i) Consider the operators $U(\theta)^*, T(\theta)^* \in \mathcal{B}(L^2(\mathbb{R}))$ defined by

$$[U(\theta)^*\psi](x) = e^{\theta/2}\psi(e^\theta x), \qquad [T(\theta)^*\psi](x) = \psi(x+\theta). \tag{10}$$

One has $U(\theta)^*U(\theta) = U(\theta)U(\theta)^* = I$ and $T(\theta)T(\theta)^* = T(\theta)^*T(\theta) = I$. Moreover, for all $\psi, \phi \in L^2(\mathbb{R})$, one obtains

$$\langle \psi | U(\theta)\phi \rangle_{L^2} = \int_{\mathbb{R}} \overline{\psi(x)}\, e^{-\theta/2}\phi(e^{-\theta}x)\, dx = \int_{\mathbb{R}} \overline{e^{\theta/2}\psi(e^\theta x)}\, \phi(x)\, dx = \langle U(\theta)^*\psi | \phi \rangle_{L^2}. \tag{11}$$

The second equality follows by the change of variables $y = e^\theta x$. Similarly,

$$\langle \psi | T(\theta)\phi \rangle = \int_{\mathbb{R}} \overline{\psi(x)}\phi(x-\theta)\, dx = \int_{\mathbb{R}} \overline{\psi(x+\theta)}\phi(x)\, dx = \langle T(\theta)^*\psi | \phi \rangle_{L^2}. \tag{12}$$

Therefore, $T(\theta)^*$ and $U(\theta)^*$ are the adjoints of $T(\theta)$ and $U(\theta)$, respectively, and hence they are unitary.

(ii) By definition, $(U(\theta)^*\phi)(x) = e^{\theta/2}\psi(e^\theta x)$. For $\phi \in C_0^\infty(\mathbb{R})$, one has

$$\left(-\frac{d^2}{dx^2}U(\theta)^*\phi\right)(x) = -e^{\theta/2}e^{2\theta}\psi''(e^\theta x). \tag{13}$$

Thus,

$$(A(\theta)\phi)(x) = e^{-\theta/2}\left(-\frac{d^2}{dx^2}U(\theta)^*\phi\right)(x) = -e^{2\theta}\phi''(e^\theta x) = \left(-e^{2\theta}\frac{d^2}{dx^2}\phi\right)(x). \tag{14}$$

Hence, for all $\phi \in C_0^\infty(\mathbb{R})$, one has $A(\theta)\phi = -e^{2\theta}\frac{d^2}{dx^2}\phi$. Since both operators are closed in the domain $H^2(\mathbb{R})$, and $C_0^\infty(\mathbb{R}) \subset H^2(\mathbb{R})$ is dense, it follows that

$$A(\theta) = -e^{2\theta}\frac{d^2}{dx^2}.$$

Similarly, $B(\theta) = -\frac{d^2}{dx^2}$.

(iii) Define $(A, \mathcal{D}) : \mathbb{C} \to \mathcal{L}[L^2(\mathbb{R})]$ by $\mathcal{D}(z) = H^2(\mathbb{R})$ and

$$A(z) := e^{2z}\left(-\frac{d^2}{dx^2}\right). \tag{15}$$

The spectrum satisfies

$$\sigma(A(z)) = e^{2z}\sigma\left(-\frac{d^2}{dx^2}\right) = \{re^{2z} : r \in [0,\infty)\}.$$

In particular, $\rho(A(z)) \neq \emptyset$. For $\psi \in H^2$, one has

$$\begin{aligned} &\|h^{-1}(A(z+h)\psi - A(z)\psi) - 2ze^{2z}\left(-\frac{d^2}{dx^2}\right)\psi\|_{L_2} \\ &\qquad \leq |h^{-1}(e^{2(z+h)} - e^{2z}) - 2ze^{2z}| \left\| \left(-\frac{d^2}{dx^2}\right)\psi \right\|_{L^2} \\ &\qquad \leq |h^{-1}(e^{2(z+h)} - e^{2z}) - 2ze^{2z}| \|\psi\|_{\mathcal{D}}. \end{aligned} \tag{16}$$

Therefore,

$$\|h^{-1}(A(z+h)-A(z))-2ze^{2z}\left(-\frac{d^2}{dx^2}\right)\|_{\mathcal{B}(D;H)} \leq |h^{-1}(e^{2(z+h)}-e^{2z})-2ze^{2z}| \to 0. \quad (17)$$

Thus, $A:\mathbb{C}\to\mathcal{B}(\mathcal{D};\mathcal{H})$ is analytic, and $(A,\mathcal{D})$ is an analytic family of type A.

(iv) One sets $B(z)=B(0)=-\frac{d^2}{dx^2}$, it is clear that $\sigma(B(z))=\sigma(B(0))=[0,\infty)$.