Technische Universität Braunschweig

4. Exercise Sheet 4

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Deadline: 20.05.2026, 13:15 Uhr

Exercise 4.1 (6)

Let

$$B := \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{C}).$$

Compute the diagonal form of B, if B is diagonalizable, or otherwise compute the Jordan normal form of B and the eigenprojections of B.

Exercise 4.2 (6)

Let $\Omega \subseteq \mathbb{C}$ be nonempty and open, and let $\mathcal{H}$ be a complex separable Hilbert space. A map

$$A : \Omega \to \mathcal{B}(\mathcal{H})$$

is called weakly analytic if, for every trace-class operator $\rho \in \mathcal{L}^1(\mathcal{H})$, the function

$$f_\rho : \Omega \to \mathbb{C}, \qquad f_\rho(z) := \mathrm{Tr}\{\rho A(z)\},$$

is analytic (in the sense of complex analysis).

Show that $A : \Omega \to \mathcal{B}(\mathcal{H})$ is analytic if and only if $A : \Omega \to \mathcal{B}(\mathcal{H})$ is weakly analytic.

Exercise 4.3 (12)

Let $\Omega \subseteq \mathbb{C}$ be open and connected with $0 \in \Omega$, let $\mathcal{H}$ be a complex separable Hilbert space, and let $\mathcal{D} \subseteq \mathcal{H}$ be a dense subspace. A mapping

$$(A, \mathcal{D}) : \Omega \to \mathcal{L}[\mathcal{H}]$$

is called an analytic family of type A if $\mathcal{D}(\theta) = \mathcal{D}$ for all $\theta \in \Omega$ and

(a) For every $\theta \in \Omega$, the operator $(A(\theta), \mathcal{D})$ is closed and $\rho[A(\theta)] \neq \emptyset$.

(b) The map

$$A : \Omega \to \mathcal{B}(\mathcal{D}; \mathcal{H})$$

is analytic, where the Hilbert space

$$(\mathcal{D}, \langle \cdot, \cdot \rangle_{\mathcal{D}})$$

is equipped with the graph inner product

$$\langle \varphi, \psi \rangle_{\mathcal{D}} := \langle A(0)\varphi, A(0)\psi \rangle + \langle \varphi, \psi \rangle.$$

Now let

$$\mathcal{H} = L^2(\mathbb{R}), \qquad \mathcal{D} := H^2(\mathbb{R}).$$

For $\theta \in \mathbb{R}$ and $\psi \in \mathcal{H}$, define

$$[U(\theta)\psi](x) := e^{-\theta/2}\psi(e^{-\theta}x)$$

and

$$[T(\theta)\psi](x) := \psi(x - \theta).$$

(i) Show that $U(\theta)$ and $T(\theta)$ are unitary for all $\theta \in \mathbb{R}$.

(ii) Compute

$$A(\theta) := U(\theta)\Big(-\frac{d^2}{dx^2}\Big)U(\theta)^*$$

and

$$B(\theta) := T(\theta)\Big(-\frac{d^2}{dx^2}\Big)T(\theta)^*$$

for every $\theta \in \mathbb{R}$.

(iii) Show that there exists $r > 0$ such that

$$A : \mathbb{R} \to \mathcal{B}(\mathcal{D};\mathcal{H})$$

admits an extension

$$A : S_r \to \mathcal{B}(\mathcal{D};\mathcal{H})$$

to an analytic family of type A on the strip

$$S_r := \mathbb{R} + i(-r,r),$$

and compute the spectrum $\sigma[A(\theta)]$ for every $\theta \in S_r$.

(iv) Show that there exists $r > 0$ such that

$$B : \mathbb{R} \to \mathcal{B}(\mathcal{D};\mathcal{H})$$

admits an extension

$$B : S_r \to \mathcal{B}(\mathcal{D};\mathcal{H})$$

to an analytic family of type A on the strip

$$S_r := \mathbb{R} + i(-r,r),$$

and compute the spectrum $\sigma[B(\theta)]$ for every $\theta \in S_r$.