Technische Universität Braunschweig

3. Exercise Sheet 3

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Let $K, N \in \mathbb{N}$, $\underline{Z} := (Z_1, \ldots, Z_K) \in (\mathbb{R}^+)^K$, $\underline{R} := (R_1, \ldots, R_K) \in (\mathbb{R}^3)^K$, $\mathfrak{h} := L^2(\mathbb{R}^3)$, $\mathfrak{H}^{(N)} := \mathfrak{h}^{\otimes N} = L^2(\mathbb{R}^{3N})$ and $\mathcal{D}^{(N)} := H^2(\mathbb{R}^{3N})$.

Exercise 3.1 (6)

Show that $\big(|\cdot|^{-1}, H^2(\mathbb{R}^3)\big) \in \mathfrak{L}[\mathfrak{h}]$ is infinitesimally bounded with respect to $\big(-\Delta, H^2(\mathbb{R}^3)\big)$.

Solution

For $R \in \mathbb{R}^3$, define $V(x) = |x-R|^{-1}$ and

$$V_2, V_\infty : \mathbb{R}^3 \to \mathbb{R}, \qquad V_2(x) = |x|^{-1}\chi_{\{x\in\mathbb{R}^3 : |x-R|<1\}}, \qquad V_\infty(x) = |x|^{-1}\chi_{\{x\in\mathbb{R}^3 : |x-R|\geq 1\}}. \tag{1}$$

Observe that $V = V_2 + V_\infty$, with $V_2 \in L^2(\mathbb{R}^3)$ and $V_\infty \in L^\infty(\mathbb{R}^3)$. Moreover,

$$\|V_2\|_{L^2} = 2\sqrt{\pi}, \qquad \|V_\infty\|_{L^\infty} = 1.$$

Since $V_\infty \in L^\infty(\mathbb{R}^3)$, it follows that for every $E > 0$, the operator $V_\infty(-\Delta + E)^{-1}$ is bounded. In addition, by Lemma III.4 of the notes, the operator $V_2(-\Delta+E)^{-1}$ is also bounded. Using Eq. (III.26) of the notes, we obtain

$$\begin{aligned} \|V(-\Delta+E)^{-1}\|_{op} &\leq \|V_2(-\Delta+E)^{-1}\|_{op} + \|V_\infty(-\Delta+E)^{-1}\|_{op} \\ &\leq CE^{-1/2} + \|V_\infty\|_{L^\infty}\,\|(-\Delta+E)^{-1}\|_{op} \\ &= CE^{-1/2} + E^{-1}, \end{aligned} \tag{2}$$

where

$$C = \sqrt{\frac{2}{3\pi}}\|V_2\|_{L^2} = 2\sqrt{\frac{2}{3}}\,\pi.$$

Therefore, as a consequence of Lemma III.3 of the notes, for every $\psi \in H^2(\mathbb{R}^{3N})$,

$$\|V\psi\|_{L^2} \leq (CE^{-1/2} + E^{-1})\|-\Delta\psi\|_{L^2} + (CE^{1/2}+1)\|\psi\|_{L^2}. \tag{3}$$

Exercise 3.2 (6)

Let $U_{\underline{Z},\underline{R}}$ be the multiplication operator on $\mathfrak{H}^{(N)}$ corresponding to the function

$$U_{\underline{Z},\underline{R}}(x_1, \ldots, x_N) \;:=\; -\sum_{n=1}^{N}\sum_{k=1}^{K}\frac{Z_k}{|x_n - R_k|} \tag{4}$$

Show that $\big(U_{\underline{Z},\underline{R}}, H^2(\mathbb{R}^{3N})\big) \in \mathfrak{L}[\mathfrak{H}^{(N)}]$ is infinitesimally bounded with respect to $\big(\sum_{n=1}^N -\Delta_n, H^2(\mathbb{R}^{3N})\big)$.

Solution.

For all $k \in \{1, \ldots, K\}$, let

$$V_k(x) = |x - R_k|^{-1}.$$

Exercise 3.1 (Eq. (3)) implies that for every $\psi \in H^2(\mathbb{R}^3)$,

$$\|V_k\psi\|_{L^2(\mathbb{R}^3)} \leq a(E)\|-\Delta\psi\|_{L^2(\mathbb{R}^3)} + b(E)\|\psi\|_{L^2(\mathbb{R}^3)}, \tag{5}$$

where

$$a(E) = CE^{-1/2} + E^{-1}, \qquad b(E) = CE^{1/2} + 1.$$

Now let $\psi \in H^2(\mathbb{R}^{3N})$ and $n \in \{1, \ldots, N\}$. For $x \in \mathbb{R}^{3(N-1)}$, define $\psi_n(x) \in L^2(\mathbb{R}^3)$ by

$$\psi_n(x)(y) = \psi(x_1, \ldots, y, x_{n+1}, \ldots, x_N).$$

Observe that

$$(-\Delta\psi_n(x))(y) = -\Delta_n\psi(x_1, \ldots, y, x_{n+1}, \ldots, x_N).$$

Using Eq. (5), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{3N}} |x_n - R_k|^{-2} |\psi(x_1, \ldots, x_N)|^2 \, dx \\
&= \int_{\mathbb{R}^{3(N-1)}} \int_{\mathbb{R}^3} |V_k(x_n)|^2 |\psi(x_1, \ldots, x_N)|^2 \, dx_n \, d^{3(N-1)}x \\
&\leq a(E)^2 \| - \Delta_n \psi\|_{L^2}^2 \\
&\quad + 2a(E)b(E) \int_{\mathbb{R}^{3(N-1)}} \| - \Delta\psi_n(x)\|_{L^2(\mathbb{R}^3)} \|\psi_n(x)\|_{L^2(\mathbb{R}^3)} \, dx \\
&\quad + b(E)^2 \|\psi\|_{L^2}^2 \\
&\leq a(E)^2 \| - \Delta_n \psi\|_{L^2(\mathbb{R}^{3N})}^2 \\
&\quad + 2a(E)b(E) \| - \Delta_n \psi\|_{L^2(\mathbb{R}^{3N})} \|\psi\|_{L^2(\mathbb{R}^{3N})} \\
&\quad + b(E)^2 \|\psi\|_{L^2(\mathbb{R}^{3N})}^2 \\
&= \left(a(E) \| - \Delta_n \psi\|_{L^2(\mathbb{R}^{3N})} + b(E) \|\psi\|_{L^2(\mathbb{R}^{3N})} \right)^2 .
\end{aligned} \tag{6}$$

Therefore, for every $\psi \in H^2(\mathbb{R}^{3N})$,

$$\|U_{\underline{Z},\underline{R}}\psi\|_{L^2(\mathbb{R}^{3N})} \leq a(E)\Sigma(Z) \sum_{n=1}^{N} \| - \Delta_n \psi\|_{L^2(\mathbb{R}^{3N})} + Nb(E)\Sigma(Z)\|\psi\|_{L^2(\mathbb{R}^{3N})}, \tag{7}$$

where $\Sigma(Z) = \sum_{k=1}^{K} Z_k$.

Integration by parts shows that for every $\psi \in H^2(\mathbb{R}^{3N})$, $\langle \partial_j^2 \psi, \partial_i^2 \psi \rangle = \|\partial_i \partial_j \psi\|^2$. Hence,

$$\begin{aligned}
\left\| \sum_{n=1}^{N} -\Delta_n \psi \right\|_{L^2}^2 &= \sum_{n=1}^{N} \| - \Delta_n \psi\|^2 + 2 \sum_{n \neq m} \langle -\Delta_n \psi, -\Delta_m \psi \rangle \\
&= \sum_{n=1}^{N} \| - \Delta_n \psi\|^2 + 2 \sum_{n \neq m} \sum_{i,j} \langle -\partial_{n,i}^2 \psi, -\partial_{m,j}^2 \psi \rangle \\
&= \sum_{n=1}^{N} \| - \Delta_n \psi\|^2 + 2 \sum_{n \neq m} \sum_{i,j} \|\partial_{n,i} \partial_{m,j} \psi\|^2 \\
&\geq \sum_{n=1}^{N} \| - \Delta_n \psi\|^2.
\end{aligned} \tag{8}$$

Using the Cauchy–Schwarz inequality together with Eq. (8), we obtain

$$\left(\sum_{n=1}^{N} \| - \Delta_n \psi\|_{L^2} \right)^2 \leq N \sum_{n=1}^{N} \| - \Delta_n \psi\|_{L^2}^2 \leq N \left\| \sum_{n=1}^{N} -\Delta_n \psi \right\|_{L^2}^2 . \tag{9}$$

Finally, combining Eqs. (9) and (7), we conclude that

$$\|U_{\underline{Z},\underline{R}}\psi\|_{L^2(\mathbb{R}^{3N})} \leq a(E)\Sigma(Z)N^{1/2}\left\|\sum_{n=1}^{N} -\Delta_n\psi\right\|_{L^2(\mathbb{R}^{3N})} + Nb(E)\Sigma(Z)\|\psi\|_{L^2(\mathbb{R}^{3N})}. \tag{10}$$

Exercise 3.3 (6)

Let V be the multiplication operator on $\mathfrak{H}^{(N)}$ corresponding to the function

$$V(x_1,\dots,x_N) \;:=\; \sum_{1\leq m<n\leq N} \frac{1}{|x_m - x_n|} \tag{11}$$

Show that $\big(V, H^2(\mathbb{R}^{3N})\big) \in \mathfrak{L}[\mathfrak{H}^{(N)}]$ is infinitesimally bounded with respect $\big(\sum_{n=1}^{N} -\Delta_n, H^2(\mathbb{R}^{3N})\big)$.

Solution. Eq. (3) together with an estimate analogous to Eq. (6) implies that for every $\psi \in H^2(\mathbb{R}^{3N})$ and every $n, m \in \{1,\dots,N\}$, $n \neq m$, one has

$$\begin{aligned} \int_{\mathbb{R}^{3N}} |x_n - x_m|^{-2}|\psi(x_1,\dots,x_N)|^2\,dx \quad &= \int_{\mathbb{R}^{3(N-1)}}\int_{\mathbb{R}^3} |x_n - x_m|^{-2}|\psi(x_1,\dots,x_N)|^2\,dx_n\,d^{3(N-1)}x \\ \leq \big(a(E)\|-\Delta_n\psi\|_{L^2} + b(E)\|\psi\|_{L^2}\big)^2, & \end{aligned} \tag{12}$$

where $a(E) = CE^{-1/2} + E^{-1}, b(E) = CE^{1/2} + 1$. Equation (12) then implies

$$\|V\psi\|_{L^2} \leq \frac{1}{2}(N-1)a(E)\sum_{n=1}^{N}\|-\Delta_n\psi\|_{L^2} + \frac{1}{2}N(N-1)b(E)\|\psi\|_{L^2}. \tag{13}$$

Finally, combining Eqs. (13) and (9), we obtain

$$\|V\psi\|_{L^2} \leq \frac{1}{2}(N-1)N^{1/2}a(E)\left\|\sum_{n=1}^{N} -\Delta_n\psi\right\|_{L^2} + \frac{1}{2}N(N-1)b(E)\|\psi\|_{L^2}. \tag{14}$$

Exercise 3.4 (6)

Show that $\big(H_N(\underline{Z},\underline{R}), H^2(\mathbb{R}^{3N})\big) \in \mathfrak{L}[\mathfrak{H}^{(N)}]$ is a semibounded self-adjoint operator, where

$$H_N(\underline{Z},\underline{R}) \;:=\; \sum_{n=1}^{N}\left(-\Delta_n - \sum_{k=1}^{K}\frac{Z_k}{|x_n - R_k|}\right) + \sum_{1\leq m<n\leq N}\frac{1}{|x_m - x_n|}\,. \tag{15}$$

Solution. Eqs. (10) and (14) imply that for every $\psi \in H^2(\mathbb{R}^{3N})$,

$$\begin{aligned} \|(U_{\underline{Z},\underline{R}} + V)\psi\| &\leq \|U_{\underline{Z},\underline{R}}\psi\| + \|V\psi\| \\ &\leq a(E)N^{1/2}\left(\Sigma(Z) + \frac{1}{2}(N-1)\right)\left\|\sum_{n=1}^{N} -\Delta_n\psi\right\|_{L^2} \\ &\quad + b(E)N\left(\Sigma(Z) + \frac{1}{2}(N-1)\right)\|\psi\|_{L^2}, \end{aligned} \tag{16}$$

where

$$a(E) = CE^{-1/2} + E^{-1}, \qquad b(E) = CE^{1/2} + 1, \qquad E > 0.$$

Therefore, $U_{\underline{Z},\underline{R}} + V$ is an infinitesimal perturbation of

$$\left(\sum_{n=1}^{N} -\Delta_n, H^2(\mathbb{R}^{3N})\right).$$

The conclusion then follows from the Kato–Rellich theorem.