2. Exercise Sheet 2

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Exercise 2.1 (3 + 3)

Let $(\mathcal{H}, \langle \cdot \mid \cdot \rangle_{\mathcal{H}})$ be a separable, infinite-dimensional complex Hilbert space with orthonormal bases $B_1 := \{\varphi_n\}_{n=1}^{\infty}$, $B_2 := \{\psi_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$, and let $A \in \mathcal{B}(\mathcal{H})$ be a bounded operator on $\mathcal{H}$.

(a) Show that

$$\|A\|_{B_1}^2 := \sum_{n=1}^{\infty} \|A\varphi_n\|^2 < \infty \quad \Longleftrightarrow \quad \|A\|_{B_2}^2 := \sum_{n=1}^{\infty} \|A\psi_n\|^2 < \infty,$$

and that in this case

$$\|A\|_{B_1} = \|A\|_{B_2}.$$

(b) Let the set of Hilbert-Schmidt operators on $\mathcal{H}$ be defined by

$$\mathcal{L}^2(\mathcal{H}) := \{A \in \mathcal{B}(\mathcal{H}) \mid \|A\|_{B_1} < \infty\}.$$

Show that

$$(\mathcal{L}^2(\mathcal{H}), \langle \cdot \mid \cdot \rangle_{\mathcal{L}^2})$$

is a separable, infinite-dimensional complex Hilbert space, where

$$\forall\, A, B \in \mathcal{L}^2(\mathcal{H}) : \quad \langle A \mid B \rangle_{\mathcal{L}^2} := \sum_{n=1}^{\infty} \langle A\varphi_n \mid B\varphi_n \rangle_{\mathcal{H}}.$$

Solution.

(a) Let us first show that $\|A\|_{B_1} < \infty$ implies $\|A^*\|_{B_1} < \infty$. Since $\{\varphi_n\}$ is an orthonormal basis, for all $\varphi \in \mathcal{H}$ one has

$$\|\varphi\|^2 = \sum_{m=1}^{\infty} |\langle \varphi_m \mid \varphi \rangle|^2. \tag{1}$$

Using Eq. (1), one obtains

$$\begin{aligned}\sum_{n=1}^{\infty} \|A^*\varphi_n\|^2 &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\langle \varphi_m \mid A^*\varphi_n \rangle|^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\overline{\langle \varphi_m \mid A^*\varphi_n \rangle}|^2 \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\langle \varphi_n \mid A\varphi_m \rangle|^2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\langle \varphi_n \mid A\varphi_m \rangle|^2 \\ &= \sum_{m=1}^{\infty} \|A\varphi_m\|^2.\end{aligned} \tag{2}$$

This proves the claim. Moreover, note that $\|A\|_{B_1} = \|A^*\|_{B_1}$. Using Eq. (1), one obtains

$$\begin{aligned}\sum_{n=1}^{\infty} \|A\psi_n\|^2 &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\langle \varphi_m \mid A\psi_n\rangle|^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\overline{\langle \varphi_m \mid A\psi_n\rangle}|^2 \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\langle \psi_n \mid A^*\varphi_m\rangle|^2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\langle \psi_n \mid A^*\varphi_m\rangle|^2 \\ &= \sum_{m=1}^{\infty} \|A^*\varphi_m\|^2. \end{aligned} \tag{3}$$

Eqs. (2) and (3) imply the result.

(b) Let us fix an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$. Let $A, B \in \mathcal{L}^2(\mathcal{H})$. Applying the Cauchy–Schwarz inequality to the sequences $\{\|A\varphi_n\|\}$ and $\{\|B\varphi_n\|\}$ (which belong to $\ell^2(\mathbb{N})$), one obtains

$$\sum_{n=1}^{\infty} \|A\varphi_n\|\,\|B\varphi_n\| \le \|A\|_{\mathcal{L}^2(\mathcal{H})}\,\|B\|_{\mathcal{L}^2(\mathcal{H})}. \tag{4}$$

Since

$$\|(A+B)\varphi_n\|^2 = \|A\varphi_n\|^2 + \langle A\varphi_n, B\varphi_n\rangle + \langle B\varphi_n, A\varphi_n\rangle + \|B\varphi_n\|^2,$$

using the Cauchy–Schwarz inequality and Eq. (4), one obtains

$$\begin{aligned}\sum_{n=1}^{\infty} \|(A+B)\varphi_n\|^2 &\le \sum_{n=1}^{\infty} \|A\varphi_n\|^2 + 2\|A\varphi_n\|\,\|B\varphi_n\| + \|B\varphi_n\|^2 \\ &\le \|A\|^2_{\mathcal{L}^2(\mathcal{H})} + 2\|A\|_{\mathcal{L}^2(\mathcal{H})}\|B\|_{\mathcal{L}^2(\mathcal{H})} + \|B\|^2_{\mathcal{L}^2(\mathcal{H})}.\end{aligned} \tag{5}$$

Eq. (5) implies that $A + B \in \mathcal{L}^2(\mathcal{H})$. Clearly, $\mathcal{L}^2(\mathcal{H})$ is closed under scalar multiplication and $0 \in \mathcal{L}^2(\mathcal{H})$. Therefore, $\mathcal{L}^2(\mathcal{H})$ is a vector subspace of $\mathcal{B}(\mathcal{H})$.

Moreover, if $\varphi \in \mathcal{H}$ and $A \in \mathcal{L}^2(\mathcal{H})$, then

$$\|A\varphi\|^2 = \sum_{n=1}^{\infty} |\langle \varphi_n \mid A\varphi\rangle|^2 = \sum_{n=1}^{\infty} |\langle A^*\varphi_n \mid \varphi\rangle|^2 \le \|\varphi\|^2 \sum_{n=1}^{\infty} \|A^*\varphi_n\|^2 = \|A\|^2_{\mathcal{L}^2(\mathcal{H})}\|\varphi\|^2. \tag{6}$$

Hence,

$$\|A\|_{\mathcal{B}(\mathcal{H})} \le \|A\|_{\mathcal{L}^2(\mathcal{H})}. \tag{7}$$

Eq. (4) also implies

$$\sum_{n=1}^{\infty} |\langle A\varphi_n \mid B\varphi_n\rangle| \le \sum_{n=1}^{\infty} \|A\varphi_n\|\,\|B\varphi_n\| \le \|A\|_{\mathcal{L}^2(\mathcal{H})}\,\|B\|_{\mathcal{L}^2(\mathcal{H})} < \infty. \tag{8}$$

Thus, the inner product $\langle \cdot \mid \cdot \rangle_{\mathcal{L}^2(\mathcal{H})}$ is well defined. It is clearly sesquilinear, positive definite, conjugate symmetric, and non-degenerate.

It remains to show that $(\mathcal{L}^2(\mathcal{H}), \|\cdot\|_{\mathcal{L}^2(\mathcal{H})})$ is complete. Let $\{A_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{L}^2(\mathcal{H})$. Eq. (7) implies that $\{A_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in $\mathcal{B}(\mathcal{H})$. Since $\mathcal{B}(\mathcal{H})$ is complete, there exists $A \in \mathcal{B}(\mathcal{H})$ such that

$$\lim_{n\to\infty} \|A_n - A\|_{\mathcal{B}(\mathcal{H})} = 0. \tag{9}$$

In particular, for all $\varphi \in \mathcal{H}$,

$$A\varphi = \lim_{n\to\infty} A_n \varphi. \tag{10}$$

We now show that $A \in \mathcal{L}^2(\mathcal{H})$ and that $\|A_n - A\|_{\mathcal{L}^2(\mathcal{H})} \to 0$. Since $\{A_n\}_{n=1}^\infty$ is a Cauchy sequence, it is bounded; hence,

$$\sup_{n\in\mathbb{N}} \|A_n\|_{\mathcal{L}^2(\mathcal{H})} < \infty. \tag{11}$$

By Fatou's lemma,

$$\sum_{k=1}^\infty \|A\varphi_k\|^2 = \sum_{k=1}^\infty \lim_{n\to\infty} \|A_n\varphi_k\|^2 \le \liminf_{n\to\infty} \sum_{k=1}^\infty \|A_n\varphi_k\|^2 \le \sup_{n\in\mathbb{N}} \|A_n\|^2_{\mathcal{L}^2(\mathcal{H})} < \infty. \tag{12}$$

Hence, $A \in \mathcal{L}^2(\mathcal{H})$.

Again by Fatou's lemma,

$$\begin{aligned} \|A - A_n\|^2_{\mathcal{L}^2(\mathcal{H})} &= \sum_{p=1}^\infty \|(A - A_n)\varphi_p\|^2 = \sum_{p=1}^\infty \lim_{m\to\infty} \|(A_m - A_n)\varphi_p\|^2 \\ &\le \liminf_{m\to\infty} \sum_{p=1}^\infty \|(A_m - A_n)\varphi_p\|^2 \\ &= \liminf_{m\to\infty} \|A_m - A_n\|^2_{\mathcal{L}^2(\mathcal{H})} \to 0, \quad n \to \infty. \end{aligned} \tag{13}$$

Therefore, $\|A_n - A\|_{\mathcal{L}^2(\mathcal{H})} \to 0$, and $(\mathcal{L}^2(\mathcal{H}), \|\cdot\|_{\mathcal{L}^2(\mathcal{H})})$ is complete.

Exercise 2.2 (3 + 3 + 3)

Let $(\mathcal{H}, \langle\cdot|\cdot\rangle_{\mathcal{H}})$ be a separable, infinite-dimensional complex Hilbert space, and let $j : \mathcal{H} \to \mathcal{H}^*$ be defined by $\psi \mapsto j[\psi]$, where $j[\psi](\varphi) = \langle\psi|\varphi\rangle$ for all $\varphi \in \mathcal{H}$.

(a) Show that $(\mathcal{H}^*, \langle\cdot|\cdot\rangle_{\mathcal{H}^*})$ with

$$\langle j[\psi_1] | j[\psi_2]\rangle_{\mathcal{H}^*} := \langle\psi_2|\psi_1\rangle_{\mathcal{H}}$$

is a complex Hilbert space and that $j : \mathcal{H} \to \mathcal{H}^*$ is an anti-unitary isomorphism.

(b) Show that

$$(\mathcal{L}^2(\mathcal{H}), \langle\cdot|\cdot\rangle_{\mathcal{L}^2})$$

is isomorphic as a Hilbert space to

$$(\mathcal{H} \otimes \mathcal{H}^*, \langle\cdot|\cdot\rangle_{\mathcal{H}\otimes\mathcal{H}^*})$$

where the inner product on $\mathcal{H} \otimes \mathcal{H}^*$ is defined by linear, continuous extension of

$$\langle\varphi_1 \otimes \psi_1 \mid \varphi_2 \otimes \psi_2\rangle_{\mathcal{H}\otimes\mathcal{H}^*} := \langle\varphi_1|\varphi_2\rangle_{\mathcal{H}} \cdot \langle\psi_1|\psi_2\rangle_{\mathcal{H}^*}.$$

(c) Let $\mathcal{H} := L^2(\mathbb{R}^3)$ and $a : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}$ be a measurable function. Define

$$[A\varphi](x) := \int a(x,y)\, \varphi(y)\, d^3y, \quad \varphi \in \mathcal{D}.$$

Show that

$$A \in \mathcal{L}^2(\mathcal{H}) \quad \Longleftrightarrow \quad a \in L^2(\mathbb{R}^3 \times \mathbb{R}^3),$$

and in this case

$$\|A\|_{\mathcal{L}^2(\mathcal{H})} = \|a\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}.$$

Solution.

(a) Let us show that j is an isomorphism. If $j[h] = 0$, then

$$\|h\|^2 = \langle h \mid h \rangle = j[h](h) = 0. \tag{14}$$

Hence $h = 0$, which proves that j is injective.

Let $\varphi \in \mathcal{H}^*$. By the Riesz representation theorem, there exists $h \in \mathcal{H}$ such that $\varphi(f) = \langle h \mid f \rangle$. Then $j[h](f) = \langle h \mid f \rangle = \varphi(f)$, which implies $j[h] = \varphi$. Thus, j is surjective. It is clear that j is linear, hence j is an isomorphism.

It is easy to see from the definition that the inner product is conjugate symmetric, sesquilinear, and positive definite. Moreover, since j is an isomorphism, the inner product is non-degenerate. Using that

$$\|j[h]\|_{\mathcal{H}^*} = \|h\|_{\mathcal{H}}, \tag{15}$$

and that j is an isomorphism, it follows that $(\mathcal{H}^*, \langle \cdot \mid \cdot \rangle_{\mathcal{H}^*})$ is complete. By definition of the inner product, one obtains

$$\langle j[h] \mid j[f] \rangle_{\mathcal{H}^*} = \langle f \mid h \rangle_{\mathcal{H}} = \overline{\langle h \mid f \rangle_{\mathcal{H}}}. \tag{16}$$

(b) For all $(h, \varphi) \in \mathcal{H} \times \mathcal{H}^*$, define the linear operator $B(h, \varphi)$ by

$$B(h,\varphi) : \mathcal{H} \to \mathcal{H}, \qquad B(h,\varphi)f = \varphi(f)h.$$

For all $f \in \mathcal{H}$ one has

$$\|B(h,\varphi)f\| = \|\varphi(f)h\| \leq \|h\| \, \|\varphi\| \, \|f\|. \tag{17}$$

Thus, $B(h,\varphi) \in \mathcal{B}(\mathcal{H})$ is bounded and $\|B(h,\varphi)\|_{op} \leq \|h\| \, \|\varphi\|$.

Moreover, for an orthonormal basis $\{h_n\}$ of $\mathcal{H}$ one has

$$\sum_{n=1}^{\infty} \|B(h,\varphi)h_n\|^2 = \sum_{n=1}^{\infty} |\varphi(h_n)|^2 \|h\|^2 = \|h\|^2 \sum_{n=1}^{\infty} |\langle j^{-1}[\varphi] \mid h_n \rangle|^2 = \|h\|^2 \|\varphi\|^2. \tag{18}$$

Hence $B(h,\varphi) \in \mathcal{L}^2(\mathcal{H})$.

Moreover, if $h, f \in \mathcal{H}$ and $\varphi, \psi \in \mathcal{H}^*$, then

$$\begin{aligned} \langle B(h,\varphi) \mid B(f,\psi) \rangle_{\mathcal{L}^2(\mathcal{H})} &= \sum_{n=1}^{\infty} \langle B(h,\varphi)h_n \mid B(f,\psi)h_n \rangle = \langle h \mid f \rangle \sum_{n=1}^{\infty} \overline{\varphi(h_n)}\psi(h_n) \\ &= \langle h \mid f \rangle \sum_{n=1}^{\infty} \langle j(h_n) \mid \psi \rangle_{\mathcal{H}^*} \langle \varphi \mid j(h_n) \rangle_{\mathcal{H}^*} \\ &= \langle h \mid f \rangle_{\mathcal{H}} \langle \varphi \mid \psi \rangle_{\mathcal{H}^*}. \end{aligned} \tag{19}$$

Consider the bilinear map

$$B : \mathcal{H} \times \mathcal{H}^* \to \mathcal{L}^2(\mathcal{H}), \qquad (h,\varphi) \mapsto B(h,\varphi). \tag{20}$$

By the universal property of the (algebraic) tensor product, there exists a linear map

$$T : \mathcal{H} \otimes_{alg} \mathcal{H}^* \to \mathcal{L}^2(\mathcal{H}) \tag{21}$$

such that $T(h \otimes_{alg} \varphi) = B(h,\varphi)$, where $\mathcal{H} \otimes_{alg} \mathcal{H}^*$ denotes the tensor product as a vector space.

Moreover, if $v = \sum_j \lambda_j h_j \otimes_{alg} \varphi_j$ and $w = \sum_i \beta_i f_i \otimes_{alg} \psi_i$, then

$$\begin{aligned}\langle T(v) \mid T(w)\rangle_{\mathcal{L}^2(\mathcal{H})} &= \sum_{i,j} \overline{\lambda_j}\beta_i \langle B(h_j, \varphi_j) \mid B(f_i, \psi_i)\rangle_{\mathcal{L}^2(\mathcal{H})} = \sum_{i,j} \overline{\lambda_j}\beta_i \langle h_j \mid f_i\rangle\langle \varphi_j \mid \psi_i\rangle \\ &= \sum_{i,j} \overline{\lambda_j}\beta_i \langle h_j \otimes \varphi_j \mid f_i \otimes \psi_i\rangle = \langle v \mid w\rangle_{\mathcal{H}\otimes\mathcal{H}^*}.\end{aligned} \tag{22}$$

Thus, T is continuous and admits a continuous extension to $\mathcal{H} \otimes \mathcal{H}^*$. Denote this extension by

$$\tilde{T} : \mathcal{H} \otimes \mathcal{H}^* \to \mathcal{L}^2(\mathcal{H}). \tag{23}$$

Eq. (22) implies that

$$\langle \tilde{T}v \mid \tilde{T}w\rangle_{\mathcal{L}^2(\mathcal{H})} = \langle v \mid w\rangle_{\mathcal{H}\otimes\mathcal{H}^*}, \qquad v, w \in \mathcal{H} \otimes \mathcal{H}^*. \tag{24}$$

In particular, $\tilde{T}$ is an isometry, hence injective. It remains to show that $\tilde{T}$ is surjective.

Let $A \in \mathcal{L}^2(\mathcal{H})$ be a Hilbert–Schmidt operator and let $\{h_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. Define

$$v := \sum_{k=1}^{\infty} Ah_k \otimes j[h_k] \in \mathcal{H} \otimes \mathcal{H}^*. \tag{25}$$

This series converges since

$$\sum_{k=1}^{\infty} \|Ah_k \otimes j[h_k]\|^2 = \sum_{k=1}^{\infty} \|Ah_k\|^2 = \|A\|^2_{\mathcal{L}^2(\mathcal{H})} < \infty. \tag{26}$$

Moreover, for all $f \in \mathcal{H}$,

$$\begin{aligned}\tilde{T}(v)f &= \sum_{k=1}^{\infty} \tilde{T}(Ah_k \otimes j[h_k])f = \sum_{k=1}^{\infty} B(Ah_k, j[h_k])f \\ &= \sum_{k=1}^{\infty} j[h_k](f)Ah_k = \sum_{k=1}^{\infty} \langle h_k \mid f\rangle Ah_k \\ &= A\sum_{k=1}^{\infty} \langle h_k \mid f\rangle h_k = Af.\end{aligned} \tag{27}$$

Thus, $\tilde{T}$ is surjective, hence an isomorphism of Hilbert spaces.

(c) $\Rightarrow$ Let N be a null set such that for all $x \in \mathbb{R}^3 \setminus N$ one has

$$a_x \in L^2(\mathbb{R}^3), \tag{28}$$

where

$$a_x(y) := a(x, y). \tag{29}$$

Fix $x \in \mathbb{R}^3 \setminus N$. Since $a_x \in L^2(\mathbb{R}^3)$ and $\{\varphi_n\}$ is an orthonormal basis, one has

$$\|a_x\|^2_{L^2(\mathbb{R}^3)} = \sum_{n=1}^{\infty} |\langle \varphi_n \mid a_x\rangle|^2, \tag{30}$$

where

$$\langle \varphi_n \mid a_x\rangle = \int_{\mathbb{R}^3} a(x, y)\overline{\varphi_n(y)}\, dy = [A\overline{\varphi_n}](x). \tag{31}$$

By the monotone convergence theorem,

$$\begin{aligned}\sum_{n=1}^{\infty} \|A\overline{\varphi_n}\|_{L^2}^2 &= \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} |[A\overline{\varphi_n}](x)|^2\,dx = \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} |\langle \varphi_n \mid a_x\rangle|^2\,dx \\ &= \int_{\mathbb{R}^3} \|a_x\|_{L^2}^2\,dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |a(x,y)|^2\,dy\,dx.\end{aligned} \tag{32}$$

Since $A \in \mathcal{L}^2(\mathcal{H})$, it follows that $a \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$.

$\Leftarrow$ If $a \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, then by Fubini's theorem $a_x \in L^2(\mathbb{R}^3)$ for a.e. $x \in \mathbb{R}^3$. Hölder's inequality implies that for all $\varphi \in L^2(\mathbb{R}^3)$,

$$|[A\varphi](x)| \leq \int_{\mathbb{R}^3} |a(x,y)\varphi(y)|\,dy \leq \|a_x\|_{L^2}\|\varphi\|_{L^2}. \tag{33}$$

Using Eq. (33), one obtains

$$\int_{\mathbb{R}^3} |[A\varphi](x)|^2\,dx \leq \|a\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}\|\varphi\|_{L^2(\mathbb{R}^3)}^2. \tag{34}$$

Thus, A is bounded. Moreover, Eq. (32) implies that $A \in \mathcal{L}^2(\mathcal{H})$ and

$$\|A\|_{\mathcal{L}^2(\mathcal{H})} = \|a\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}.$$

Exercise 2.3 (3 + 3 + 3)

Let $(\mathcal{H}, \langle\cdot|\cdot\rangle_{\mathcal{H}})$ be a separable, infinite-dimensional complex Hilbert space.

(a) Prove that $\mathcal{L}^2(\mathcal{H}) \subseteq \mathrm{Com}(\mathcal{H})$ holds.

(b) Let $\mathcal{H} := L^2(\mathbb{R}^d)$ and $V \in L^2(\mathbb{R}^d)$ with $d \in \{1,2,3\}$. Compute the integral kernel a of $A := V(-\Delta+1)^{-1} : \mathcal{H} \to \mathcal{H}$ and show that $A \in \mathcal{L}^2(\mathcal{H})$.

(c) Deduce that $\sigma_{\mathrm{ess}}(-\Delta+V) = \mathbb{R}_0^+$.

Solutions

(a) Let $A \in \mathcal{L}^2(\mathcal{H})$. We show that A is the limit of finite-range operators. Let $\{\varphi_n\}$ be an orthonormal basis of $\mathcal{H}$, and consider the sequence of operators

$$A_n : \mathcal{H} \to \mathcal{H}, \qquad A_n\varphi := \sum_{k=1}^{n} \langle \varphi_k \mid A\varphi\rangle \varphi_k, \quad n \in \mathbb{N}. \tag{35}$$

By definition,

$$\mathrm{Ran}(A_n) = \langle \{\varphi_k : k = 1, \ldots, n\}\rangle,$$

hence A_n is a finite-range operator. Moreover,

$$(A - A_n)\varphi = \sum_{k=1}^{\infty} \langle \varphi_k \mid A\varphi\rangle\varphi_k - \sum_{k=1}^{n} \langle \varphi_k \mid A\varphi\rangle\varphi_k = \sum_{k=n+1}^{\infty} \langle \varphi_k \mid A\varphi\rangle\varphi_k. \tag{36}$$

Using Eq. (36) and the Cauchy–Schwarz inequality, one obtains

$$\|(A - A_n)\varphi\|^2 = \sum_{k=n+1}^{\infty} |\langle \varphi_k \mid A\varphi\rangle|^2 \leq \|\varphi\|^2 \sum_{k=n+1}^{\infty} \|A^*\varphi_k\|^2. \tag{37}$$

This estimate, together with the fact that $A^* \in \mathcal{L}^2(\mathcal{H})$, implies

$$\|A - A_n\|_{\mathcal{B}(\mathcal{H})}^2 \leq \sum_{k=n+1}^{\infty} \|A^*\varphi_k\|^2 \to 0, \quad n \to \infty.$$

(b) Let

$$\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

denote the unitary extension of the Fourier transform. For all $f \in L^2(\mathbb{R}^d)$, one has

$$\mathcal{F}(-\Delta + 1)^{-1}\mathcal{F}^* f = G \cdot f, \tag{38}$$

where $G : \mathbb{R}^d \to \mathbb{R}$ is given by

$$G(x) = \frac{1}{\|x\|^2 + 1}. \tag{39}$$

Note that for $d = 1, 2, 3$ one has $G \in L^2(\mathbb{R}^d)$. For $f \in L^2(\mathbb{R}^d)$, we write

$$V(-\Delta + 1)^{-1} f = V\mathcal{F}^*\mathcal{F}(-\Delta + 1)^{-1}\mathcal{F}^*\mathcal{F} f = V\mathcal{F}^*(G \cdot (\mathcal{F} f)). \tag{40}$$

Since $G \in L^2(\mathbb{R}^d)$, one has $G \cdot (\mathcal{F} f) \in L^1(\mathbb{R}^d)$, and thus $\mathcal{F}^*(G \cdot (\mathcal{F} f)) \in L^\infty(\mathbb{R}^d)$. Moreover,

$$\|V(-\Delta + 1)^{-1} f\|_{L^2} \le \|V\|_{L^2} \|\mathcal{F}^*(G \cdot (\mathcal{F} f))\|_{L^\infty} \le \|V\|_{L^2} \|G\|_{L^2} \|f\|_{L^2}. \tag{41}$$

Hence, $V(-\Delta + 1)^{-1}$ is a bounded operator.

Moreover, for all $f, g \in L^2(\mathbb{R}^d)$ one has the general identity

$$\mathcal{F}^*(fg) = \mathcal{F}^*(f) * \mathcal{F}^*(g). \tag{42}$$

Indeed, consider the maps

$$T_1, T_2 : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d), \quad T_1(f, g) = f * g, \quad T_2(f, g) = \mathcal{F}^*(\mathcal{F}(f)\mathcal{F}(g)).$$

Both T_1 and T_2 are continuous, and by the convolution theorem they coincide on the dense subspace $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$. Hence $T_1 = T_2$, which proves (42).

Using Eqs. (40) and (42), one obtains

$$(V(-\Delta + 1)^{-1} f)(x) = V(x)((\mathcal{F}^* G) * f)(x) = \int_{\mathbb{R}^d} V(x)(\mathcal{F}^* G)(x - y) f(y)\, dy. \tag{43}$$

Thus, $V(-\Delta + 1)^{-1}$ is an integral operator with kernel

$$a(x, y) = V(x)(\mathcal{F}^* G)(x - y).$$

Since $G \in L^2(\mathbb{R}^d)$, one has

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |a(x, y)|^2\, dx\, dy = \int_{\mathbb{R}^d} |V(x)|^2\, dx \int_{\mathbb{R}^d} |(\mathcal{F}^* G)(y)|^2\, dy = \|V\|_{L^2}^2 \|G\|_{L^2}^2 < \infty. \tag{44}$$

Hence $V(-\Delta + 1)^{-1} \in \mathcal{L}^2(\mathcal{H})$.

(c) By parts (a) and (b), one has $V(-\Delta + 1)^{-1} \in \mathrm{Com}(\mathcal{H})$. Moreover, $H^2(\mathbb{R}^d) \subset \mathcal{D}_V$. Indeed, if $f \in H^2(\mathbb{R}^d)$, then $W\mathcal{F} f \in L^2(\mathbb{R}^d)$ where $W(x) = 1 + \|x\|^2$. Since $W^{-1} \in L^2(\mathbb{R}^d)$ for $d = 1, 2, 3$, one has

$$\mathcal{F} f = W^{-1} \cdot (W\mathcal{F} f) \in L^1(\mathbb{R}^d).$$

Hence $f = \mathcal{F}^*(\mathcal{F} f) \in L^\infty(\mathbb{R}^d)$. Therefore,

$$H^2(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d) \subset \mathcal{D}_V.$$

Since $V : \mathbb{R}^d \to \mathbb{R}$ is symmetric, Theorem III.8 in the notes implies

$$[0, \infty) = \sigma_{ess}(-\Delta) = \sigma_{ess}(-\Delta + V). \tag{45}$$