Technische Universität Braunschweig

2. Exercise Sheet 2

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Deadline: 29.04.2026, 13:15 Uhr

Exercise 2.1 (3 + 3)

Let $(\mathcal{H}, \langle \cdot \mid \cdot \rangle_{\mathcal{H}})$ be a separable, infinite-dimensional complex Hilbert space with orthonormal bases $B_1 := \{\varphi_n\}_{n=1}^{\infty}$, $B_2 := \{\psi_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$, and let $A \in \mathcal{B}(\mathcal{H})$ be a bounded operator on $\mathcal{H}$.

(a) Show that

$$\|A\|_{B_1}^2 := \sum_{n=1}^{\infty} \|A\varphi_n\|^2 < \infty \iff \|A\|_{B_2}^2 := \sum_{n=1}^{\infty} \|A\psi_n\|^2 < \infty,$$

and that in this case

$$\|A\|_{B_1} = \|A\|_{B_2}.$$

(b) Let the set of Hilbert-Schmidt operators on $\mathcal{H}$ be defined by

$$\mathcal{L}^2(\mathcal{H}) := \{A \in \mathcal{B}(\mathcal{H}) \mid \|A\|_{B_1} < \infty\}.$$

Show that

$$(\mathcal{L}^2(\mathcal{H}), \langle \cdot \mid \cdot \rangle_{\mathcal{L}^2})$$

is a separable, infinite-dimensional complex Hilbert space, where

$$\forall\, A, B \in \mathcal{L}^2(\mathcal{H}) : \quad \langle A \mid B \rangle_{\mathcal{L}^2} := \sum_{n=1}^{\infty} \langle A\varphi_n \mid B\varphi_n \rangle_{\mathcal{H}}.$$

Exercise 2.2 (3 + 3 + 3)

Let $(\mathcal{H}, \langle \cdot | \cdot \rangle_{\mathcal{H}})$ be a separable, infinite-dimensional complex Hilbert space, and let $j : \mathcal{H} \to \mathcal{H}^*$ be defined by $\psi \mapsto j[\psi]$, where $j[\psi](\varphi) = \langle \psi | \varphi \rangle$ for all $\varphi \in \mathcal{H}$.

(a) Show that $(\mathcal{H}^*, \langle \cdot | \cdot \rangle_{\mathcal{H}^*})$ with

$$\langle j[\psi_1] | j[\psi_2] \rangle_{\mathcal{H}^*} := \langle \psi_2 | \psi_1 \rangle_{\mathcal{H}}$$

is a complex Hilbert space and that $j : \mathcal{H} \to \mathcal{H}^*$ is an anti-unitary isomorphism.

(b) Show that

$$(\mathcal{L}^2(\mathcal{H}), \langle \cdot | \cdot \rangle_{\mathcal{L}^2})$$

is isomorphic as a Hilbert space to

$$(\mathcal{H} \otimes \mathcal{H}^*, \langle \cdot | \cdot \rangle_{\mathcal{H} \otimes \mathcal{H}^*})$$

where the inner product on $\mathcal{H} \otimes \mathcal{H}^*$ is defined by linear, continuous extension of

$$\langle \varphi_1 \otimes \psi_1 \mid \varphi_2 \otimes \psi_2 \rangle_{\mathcal{H} \otimes \mathcal{H}^*} := \langle \varphi_1 | \varphi_2 \rangle_{\mathcal{H}} \cdot \langle \psi_1 | \psi_2 \rangle_{\mathcal{H}^*}.$$

(c) Let $\mathcal{H} := L^2(\mathbb{R}^3)$ and $a : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}$ be a measurable function. Define

$$[A\varphi](x) := \int a(x,y)\, \varphi(y)\, d^3y, \quad \varphi \in \mathcal{H},$$

and show that

$$A \in \mathcal{L}^2(\mathcal{H}) \quad \Longleftrightarrow \quad a \in L^2(\mathbb{R}^3 \times \mathbb{R}^3),$$

and in this case

$$\|A\|_{\mathcal{L}^2(\mathcal{H})} = \|a\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}.$$

Exercise 2.3 (3 + 3 + 3)

Let $(\mathcal{H}, \langle \cdot | \cdot \rangle_{\mathcal{H}})$ be a separable, infinite-dimensional complex Hilbert space.

(a) Prove that $\mathcal{L}^2(\mathcal{H}) \subseteq \mathrm{Com}(\mathcal{H})$ holds.

(b) Let $\mathcal{H} := L^2(\mathbb{R}^d)$ and $V \in L^2(\mathbb{R}^d)$ with $d \in \{1,2,3\}$. Compute the integral kernel a of $A := V(-\Delta + 1)^{-1} : \mathcal{H} \to \mathcal{H}$ and show that $A \in \mathcal{L}^2(\mathcal{H})$.

(c) Deduce that $\sigma_{\mathrm{ess}}(-\Delta + V) = \mathbb{R}_0^+$.