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1. Exercise Sheet 1

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Exercise 1.1 (3 + 3 Points)

Let $\mathfrak{H}$ be a (complex, separable) Hilbert space and let $(A, \mathcal{D}) \in \mathfrak{L}[\mathfrak{H}]$ be a densely defined, symmetric (linear) operator on $\mathfrak{H}$.

(a) Show that $(A, \mathcal{D})$ is closable.

(b) Show that the following holds true

$$\forall\, \varepsilon > 0\,,\ \varphi \in \mathcal{D}: \quad \|(A - i\varepsilon)\varphi\| \ \geq\ \varepsilon\,\|\varphi\|. \tag{1}$$

Solution.

(a) Since $(A, \mathcal{D})$ is symmetric, one has

$$(A, \mathcal{D}) \subset (A^*, \mathcal{D}^*).$$

Indeed, if $h \in \mathcal{D}$, then for all $f \in \mathcal{D}$ one has

$$\langle h | Af \rangle = \langle Ah | f \rangle, \tag{2}$$

which implies that $h \in \mathcal{D}^*$ and $A^* h = Ah$. Hence, $(A^*, \mathcal{D}^*)$ is an extension of $(A, \mathcal{D})$. Thus, it is enough to show that $(A^*, \mathcal{D}^*)$ is closed. Let $\phi_n \in \mathcal{D}^*$ be such that

$$\|\phi_n - \phi\|_{\mathfrak{H}} \to 0, \qquad \|A^*\phi_n - \psi\|_{\mathfrak{H}} \to 0 \tag{3}$$

for some $\phi, \psi \in \mathfrak{H}$. Then, for all $h \in \mathcal{D}$, one has

$$\langle \phi | Ah \rangle = \lim_{n\to\infty} \langle \phi_n | Ah \rangle = \lim_{n\to\infty} \langle A^*\phi_n | h \rangle = \langle \psi | h \rangle. \tag{4}$$

Therefore, $\phi \in \mathcal{D}^*$ and $A^*\phi = \psi$.

(b) Let $\varepsilon > 0$, and let $\varphi \in \mathcal{D}$. Since A is symmetric,

$$\langle \varphi | A\varphi \rangle = \langle A\varphi | \varphi \rangle.$$

Then one has

$$\begin{aligned} \|(A - i\varepsilon)\varphi\|^2 &= \langle (A - i\varepsilon)\varphi | (A - i\varepsilon)\varphi \rangle \\ &= \|A\varphi\|^2 + i\varepsilon\langle \varphi | A\varphi \rangle - i\varepsilon \langle A\varphi | \varphi \rangle + \varepsilon^2 \|\varphi\|^2 \\ &= \|A\varphi\|^2 + \varepsilon^2\|\varphi\|^2 \\ &\geq \varepsilon^2 \|\varphi\|^2. \end{aligned} \tag{5}$$

Exercise 1.2 (3 + 3 points)

Let $\mathfrak{H} = L^2(\mathbb{R})$ and let $A : L^1(\mathbb{R}) \to L^2(\mathbb{R})$ be defined as

$$\forall\, \psi \in L^1(\mathbb{R}),\ x \in \mathbb{R} \quad [A\psi](x) \ :=\ e^{-x^2} \int_{-\infty}^{\infty} \psi(y)\, dy \tag{6}$$

(a) Provide a domain $\mathcal{D} \subseteq \mathfrak{H}$ such that $(A, \mathcal{D}) \in \mathfrak{L}[\mathfrak{H}]$ is a densely defined operator on $\mathfrak{H}$.

(b) Is $(A, \mathcal{D})$ closable?

Solution.

(a) Let us consider $\mathcal{D} := L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. By defintion, $(A, \mathcal{D}) \in \mathfrak{L}[\mathfrak{H}]$. Moreover, since $C_0(\mathbb{R})$, the compact supported continuous functions, are contained in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $C_0(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ one obtains that $\mathcal{D}$ is dense.

(b) $(A, \mathcal{D})$ is not closable. Let us prove the last statement. Let

$$\phi_n := \frac{1}{n} 1_{[0,n]}.$$

Note that $\phi_n \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then $\phi_n \oplus A\phi_n \in \mathcal{G}(A, \mathcal{D})$ and

$$[A\phi_n](x) = e^{-x^2} \int_{-\infty}^{\infty} \phi_n(y) dy = e^{-x^2} \frac{1}{n} \int_0^n 1 dy = e^{-x^2} =: F(x). \tag{7}$$

Moreover,

$$\|\phi_n\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |\phi_n|^2 = \frac{1}{n^2} \int_0^n 1 = \frac{1}{n}. \tag{8}$$

Eqs. (7) and (8) imply

$$\|\phi_n \oplus A\phi_n - 0 \oplus F\|^2_{\mathfrak{H} \oplus \mathfrak{H}} = \|\phi_n\|^2_{\mathfrak{H}} + \|A\phi_n - F\|^2_{\mathfrak{H}} = 1/n \to 0. \tag{9}$$

Then $0 \oplus F \in \overline{\mathcal{G}(A, \mathcal{D})}$. Since $F \neq 0$, (recal $F(x) := e^{-x^2}$), $\overline{\mathcal{G}(A, \mathcal{D})}$ can not be the graph of a linear operator.

Exercise 1.3 (4 + 4 + 4 points)

Let $\mathfrak{H} := L^2(\mathbb{R})$ and let $f : \mathbb{R} \to \mathbb{R}$ be a measurable real-valued function,

$$\mathcal{D} := \left\{ \varphi \in \mathfrak{H} \;\middle|\; f \cdot \varphi \in \mathfrak{H} \right\} \tag{10}$$

and let $A : \mathcal{D} \to \mathfrak{H}$ be the multiplication operator defined as follows,

$$\forall\, x \in \mathbb{R} \quad [A\varphi](x) := f(x) \cdot \varphi(x). \tag{11}$$

(a) Show that $(A, \mathcal{D})$ is self-adjoint.

(b) Provide a counterexample to the conjecture that, under the above assumptions,

$$\sigma(A) = f(\mathbb{R}) \tag{12}$$

holds. Here, the spectrum $\sigma(A) \subseteq \mathbb{C}$ of A is the set of complex numbers $z \in \mathbb{C}$ for which $A - z \cdot \mathbf{1}$ does not have a bounded inverse.

(c) Show that the spectrum $\sigma(A) \subseteq \mathbb{C}$ of A is given by

$$\sigma(A) = \bigcap \left\{ \overline{f(M)} \;\middle|\; M \in \mathfrak{L}_1, \mathbb{R} \setminus M \text{ is a Lebesgue null set} \right\} \tag{13}$$

Solution.

(a) Let us first show that $(A, \mathcal{D})$ is a densely defined operator. Consider $\varphi \in L^2(\mathbb{R})$. For all $n \in \mathbb{N}$, let us set

$$\varphi_n := 1_{f^{-1}([-n,n])}\varphi. \tag{14}$$

Since $|\varphi_n| \le |\varphi|$, it follows that $\varphi_n \in L^2(\mathbb{R})$. Moreover, one has

$$\int_{\mathbb{R}} |f\varphi_n|^2 = \int_{\mathbb{R}} |f|^2|\varphi|^2 1_{f^{-1}([-n,n])} \le n^2 \int_{\mathbb{R}} |\varphi|^2 < \infty. \tag{15}$$

Hence, $f \cdot \varphi_n \in L^2(\mathbb{R})$, and therefore $\varphi_n \in \mathcal{D}$. Since $|\varphi_n - \varphi|^2 \le 4|\varphi|^2 \in L^1(\mathbb{R})$, the Dominated Convergence Theorem implies

$$\lim_{n\to\infty} \int_{\mathbb{R}} |\varphi_n - \varphi|^2 = \int_{\mathbb{R}} \lim_{n\to\infty} |\varphi_n - \varphi|^2 = 0. \tag{16}$$

Thus,

$$\|\varphi_n - \varphi\|_{L^2(\mathbb{R})} \to 0.$$

We conclude that $\mathcal{D}$ is dense, and therefore $(A, \mathcal{D})$ is densely defined. Moreover, $(A, \mathcal{D})$ is symmetric. Indeed, if $\varphi, \psi \in \mathcal{D}$, then (recall that $f(x) \in \mathbb{R}$)

$$\langle \varphi | A\psi \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \overline{\varphi}\, A\psi = \int_{\mathbb{R}} \overline{\varphi}\, f \cdot \psi = \int_{\mathbb{R}} \overline{f \cdot \varphi}\, \psi = \langle A\varphi | \psi \rangle_{L^2(\mathbb{R})}. \tag{17}$$

Hence, $\mathcal{D} \subset \mathcal{D}^*$. In order to show that $(A, \mathcal{D})$ is selfadjoint, it remains to prove that $\mathcal{D}^* \subset \mathcal{D}$. Let us first note that $\mathrm{Ran}(A \pm i) = L^2(\mathbb{R})$. Since $f(\mathbb{R}) \subset \mathbb{R}$, one has

$$|f \pm i| = \sqrt{f^2 + 1} \ge \max(|f|, 1). \tag{18}$$

Let $h \in L^2(\mathbb{R})$. Note that

$$|h \cdot (f \pm i)^{-1}| \le |h| \in L^2(\mathbb{R}), \qquad |f \cdot h \cdot (f \pm i)^{-1}| = |h|\,|f|\,|f \pm i|^{-1} \le |h| \in L^2(\mathbb{R}). \tag{19}$$

Thus, $h \cdot (f \pm i)^{-1} \in \mathcal{D}$, and

$$\begin{aligned}(A + i)(h \cdot (f + i)^{-1}) &= A(h \cdot (f + i)^{-1}) + ih \cdot (f + i)^{-1} \\ &= f \cdot h \cdot (f + i)^{-1} + ih \cdot (f + i)^{-1} \\ &= h \cdot (f + i)^{-1} \cdot (f + i) = h.\end{aligned} \tag{20}$$

Similarly, $(A - i)(h \cdot (f - i)^{-1}) = h$. This proves that $\mathrm{Ran}(A \pm i) = L^2(\mathbb{R})$. Let us now show that $\mathcal{D}^* \subset \mathcal{D}$. Take $\varphi \in \mathcal{D}^*$ and consider $g \in \mathcal{D}$ such that

$$(A^* + i)\varphi = (A + i)g.$$

Let $h \in \mathfrak{H}$, and choose $\psi \in \mathcal{D}$ such that $(A - i)\psi = h$. Then one has

$$\langle \varphi | h \rangle = \langle \varphi | (A - i)\psi \rangle = \langle (A^* + i)\varphi | \psi \rangle = \langle (A + i)g | \psi \rangle = \langle g | (A - i)\psi \rangle = \langle g | h \rangle. \tag{21}$$

Since the above holds for all $h \in \mathfrak{H}$, one concludes that $\varphi = g \in \mathcal{D}$.

(b) Let us define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x, \quad x \neq 0, \qquad f(0) = 1.$$

Note that $f(\mathbb{R}) = \mathbb{R} \setminus \{0\}$. However, $0 \in \sigma(A)$. Indeed, if $0 \notin \sigma(A)$, then there exists $B \in \mathcal{B}(L^2(\mathbb{R}))$ bounded such that

$$AB\phi = \phi, \qquad \forall \phi \in L^2(\mathbb{R}).$$

In particular, if $\phi = 1_{[-1,1]}$, then $\phi \in L^2(\mathbb{R})$, $B\phi \in L^2(\mathbb{R})$, and

$$\phi(x) = (AB\phi)(x) = f(x) \cdot (B\phi)(x), \qquad \text{a.e.} \tag{22}$$

Hence,

$$(B\phi)(x) = \phi(x)x^{-1} = 1_{[-1,1]}(x)x^{-1}, \qquad \text{a.e.}$$

This is impossible, since the map $\mathbb{R} \ni x \mapsto 1_{[-1,1]}(x)x^{-1}$ is not square integrable.

(c) $\supset$ Suppose that

$$E \in \bigcap \left\{ \overline{f(M)} \;\middle|\; M \in \mathfrak{L}_1,\ \mathbb{R} \setminus M \text{ is a Lebesgue null set} \right\}.$$

Assume, in order to obtain a contradiction, that $E \notin \sigma(A)$. Then there exists $B \in \mathcal{B}(L^2(\mathbb{R}))$ bounded such that

$$(A - E)B\phi = \phi, \qquad \phi \in L^2(\mathbb{R}). \tag{23}$$

Let $C > 0$ and consider

$$D_C := \{x \in \mathbb{R} : |f(x) - E| < C^{-1}\}.$$

Let us prove that

$$\mu(D_C) > 0. \tag{24}$$

If $\mu(D_C) = 0$, then by assumption

$$E \in \overline{f(\mathbb{R} \setminus D_C)} \subset (-\infty, E - C^{-1}] \cup [E + C^{-1}, \infty),$$

which is impossible. Hence, (24) holds.

For all $n \in \mathbb{Z}$, let us set

$$\phi_n := 1_{[n,n+1) \cap D_C} \in L^2(\mathbb{R}).$$

Equation (23) implies that there exists a Lebesgue null set N such that

$$(f(x) - E)(B\phi_n)(x) = \phi_n(x), \qquad x \in \mathbb{R} \setminus N, \quad n \in \mathbb{Z}. \tag{25}$$

In particular,

$$|(B\phi_n)(x)| \geq C|\phi_n(x)|, \qquad x \in D_C \setminus N, \quad n \in \mathbb{Z}. \tag{26}$$

Since $\mu(D_C \setminus N) > 0$, there exists $n \in \mathbb{Z}$ such that

$$\mu(([n, n+1) \cap D_C) \setminus N) > 0.$$

Using (26), one obtains

$$\begin{aligned} \|B\phi_n\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} |(B\phi_n)(x)|^2 dx \\ &\geq \int_{(D_C \cap [n,n+1)) \setminus N} |(B\phi_n)(x)|^2 dx \\ &\geq C \int_{D_C \cap [n,n+1)} |\phi_n(x)|^2 \, dx \\ &= C\|\phi_n\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{27}$$

Since $\phi_n \neq 0$, it follows that
$$\|B\| \geq \frac{\|B\phi_n\|_{L^2(\mathbb{R})}}{\|\phi_n\|_{L^2(\mathbb{R})}} \geq \sqrt{C}. \tag{28}$$
Since this holds for all $C > 0$, we obtain a contradiction with the boundedness of B.

$\subset$ Suppose that
$$E \notin \bigcap \left\{ \overline{f(M)} \;\middle|\; M \in \mathfrak{L}_1,\, \mathbb{R} \setminus M \text{ is a Lebesgue null set} \right\}.$$
Then there exists a Lebesgue null set N such that $E \notin \overline{f(M)}$, where $M := \mathbb{R} \setminus N$. In particular,
$$\inf\{|f(x) - E| : x \in M\} = C > 0. \tag{29}$$
Consider the multiplication operator
$$T : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \qquad (Tg)(x) = \frac{1}{f(x) - E}\, g(x), \quad x \in M.$$
Note that T is well defined. Indeed, for $g \in L^2(\mathbb{R})$ one has
$$\int_{\mathbb{R}} \frac{1}{|f(x) - E|^2} |g(x)|^2 \, dx \leq C^{-2} \|g\|^2_{L^2(\mathbb{R})}. \tag{30}$$
Hence, $T \in \mathcal{B}(L^2(\mathbb{R}))$ and $\|T\| \leq C^{-1}$. Moreover, if $g \in L^2(\mathbb{R})$, then
$$|f \cdot Tg| \leq \big(1 + |(f - E)^{-1}||E|\big)|g| \leq (1 + C^{-1}|E|)|g| \in L^2(\mathbb{R}). \tag{31}$$
Thus, $Tg \in \mathcal{D}$. Clearly,
$$T(A - E)g = g, \qquad g \in \mathcal{D},$$
and
$$(A - E)Tg = g, \qquad g \in L^2(\mathbb{R}).$$
Therefore, $E \notin \sigma(A)$.