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11. Übungsblatt

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Aufgabe 11.1 (2 + 3 + 3)

Seien $\mathbb{S}^2 = (S^2, \mathfrak{T}_{rel}, \mathcal{A})$ die 2-Sphäre mit $S^2 := \{q \in \mathbb{R}^3 : \|q\|_{\text{eukl}} = 1\}$. Betrachte die Karten auf $\mathbb{S}^2$,

$$\phi_N = (S^2 \setminus \{(0,0,1)\}, x^N), \quad \phi_S = (S^2 \setminus \{(0,0,-1)\}, x^S),$$

gegeben durch die stereografische Projektion, d. h.

$$x^N(a,b,c) = \frac{1}{1-c}(a,b), \quad x^S(a,b,c) = \frac{1}{1+c}(a,b).$$

(a) Für alle $q \in S^2 \setminus \{(0,0,1),(0,0,-1)\}$ berechne $dx_1^N(q), dx_2^N(q)$ in Abhängigkeit von $dx_1^S(q), dx_2^S(q)$.

(b) Berechne $dx_1^N(q) \wedge dx_2^N(q)$ in Abhängigkeit von $dx_1^S(q) \wedge dx_2^S(q)$.

(c) Zeige, dass es eine 2-Form auf $\mathbb{S}^2$, $\omega \in \bigwedge^2[\mathbb{S}^2]$, gibt, sodass

$$\omega(q) = -(1-c)^2\, dx_1^N(q) \wedge dx_2^N(q), \quad q = (a,b,c) \in S^2 \setminus \{(0,0,1)\}, \tag{1}$$

und

$$\omega(q) = (1+c)^2\, dx_1^S(q) \wedge dx_2^S(q), \quad q = (a,b,c) \in S^2 \setminus \{(0,0,-1)\}. \tag{2}$$

Solution. Let M be a smooth manifold and let $\phi = (U,x)$ and $\psi = (V,y)$ be coordinate charts such that $q \in U \cap V$. Then,

$$dx_i(q) = (\Theta^*_{\phi,q})^{-1}(e_i) = \sum_{j=1}^n \alpha_{i,j} (\Theta^*_{\psi,q})^{-1}(e_j) = \sum_{j=1}^n \alpha_{i,j}\, dy_j(q). \tag{3}$$

Applying the operator $\Theta^*_{\psi,q}$ to both sides of the above equality, we obtain (recall that $\Theta^*_{\psi,q} \circ (\Theta^*_{\phi,q})^{-1} v = v\, J_{x \circ y^{-1}}(y(q))$)

$$\sum_{j=1}^n \alpha_{i,j} e_j = e_i\, J_{x\circ y^{-1}}(y(q)). \tag{4}$$

This implies that

$$\alpha_{i,j} = J_{x\circ y^{-1}}(y(q))_{i,j}.$$

(a) Let $q = (a,b,c) \in \mathbb{S}^2 \setminus \{(0,0,1),(0,0,-1)\}$. Then

$$dx_1^N(q) = \alpha_{11}(q)\, dx_1^S(q) + \alpha_{12}(q)\, dx_2^S(q), \qquad dx_2^N(q) = \alpha_{21}(q)\, dx_1^S(q) + \alpha_{22}(q)\, dx_2^S(q),$$

where $\alpha_{ij} = J_{x^N \circ (x^S)^{-1}}(x^S(q))$. We know that

$$x^N \circ (x^S)^{-1}(a,b) = \frac{1}{a^2+b^2}(a,b).$$

Therefore,

$$J_{x^N\circ(x^S)^{-1}}(a,b) = \frac{1}{(a^2+b^2)^2}\begin{pmatrix} b^2-a^2 & -2ab \\ -2ab & a^2-b^2 \end{pmatrix}. \tag{5}$$

Consequently,

$$\begin{aligned}
J_{x^N\circ(x^S)^{-1}}(x^S(a,b,c)) &= J_{x^N\circ(x^S)^{-1}}\left(\frac{a}{c+1},\frac{b}{c+1}\right)\\
&= \frac{(c+1)^2}{(a^2+b^2)^2}\begin{pmatrix} b^2-a^2 & -2ab\\ -2ab & a^2-b^2\end{pmatrix}\\
&= \frac{1}{(1-c)^2}\begin{pmatrix} b^2-a^2 & -2ab\\ -2ab & a^2-b^2\end{pmatrix}.
\end{aligned} \tag{6}$$

Hence,

$$-\alpha_{22}(a,b,c)=\alpha_{11}(a,b,c)=\frac{b^2-a^2}{(1-c)^2},\qquad \alpha_{12}(a,b,c)=\alpha_{21}(a,b,c)=-\frac{2ab}{(c-1)^2}. \tag{7}$$

(b) We compute

$$\begin{aligned}
dx_1^N(q)\wedge dx_2^N(q) &= (\alpha_{11}(q)\,dx_1^S(q)+\alpha_{12}(q)\,dx_2^S(q))\wedge(\alpha_{21}(q)\,dx_1^S(q)+\alpha_{22}(q)\,dx_2^S(q))\\
&= (\alpha_{11}(q)\alpha_{22}(q)-\alpha_{12}(q)\alpha_{21}(q))\,dx_1^S(q)\wedge dx_2^S(q)\\
&= \det\big(J_{x^N\circ(x^S)^{-1}}(x^S(a,b,c))\big)\,dx_1^S(q)\wedge dx_2^S(q)\\
&= -\frac{(a^2+b^2)^2}{(1-c)^4}\,dx_1^S(q)\wedge dx_2^S(q)\\
&= -\frac{(1+c)^2}{(1-c)^2}\,dx_1^S(q)\wedge dx_2^S(q).
\end{aligned} \tag{8}$$

(c) Let $\{\phi_N,\phi_S\}$ be a partition of unity subordinated to the open cover $\{\mathbb{S}^2\setminus\{(0,0,1)\},\,\mathbb{S}^2\setminus\{(0,0,-1)\}\}$. Define the differential form $\omega\in\bigwedge^2(\mathbb{S}^2)$ by, for $q=(a,b,c)\in\mathbb{S}^2$,

$$\omega(q)=-\phi_N(q)(1-c)^2\,dx_1^N(q)\wedge dx_2^N(q)+\phi_S(q)(1+c)^2\,dx_1^S(q)\wedge dx_2^S(q). \tag{9}$$

Using Eq. (8) and $\phi_N+\phi_S=1$ one see that ω satisfies the desired.

Aufhabe 11.2 (2 + 3 + 3)

Sei $(\mathbb{R}^{2n},\mathfrak{T}_{\mathbb{R}^{2n}},\mathcal{A}_{\mathbb{R}^{2n}})$ der $\mathbb{R}^{2n}$-Raum mit dem üblichen Atlas. Betrachte die Karte

$$\phi=(\mathbb{R}^{2n},\mathrm{id}),$$

die Identitätskarte auf $\mathbb{R}^{2n}$. Sei

$$\{dx_1(q),\dots,dx_n(q),dy_1(q),\dots,dy_n(q)\}\subset T^*[\mathbb{R}^{2n}]$$

die Basis des Kotangentialraums $T^*[\mathbb{R}^{2n}]$, definiert durch die Karte ϕ. Betrachte die folgende 2-Form auf $\mathbb{R}^{2n}$,

$$\alpha\in\bigwedge^2[\mathbb{R}^{2n}]\subset T_0^2[\mathbb{R}^{2n}],$$

definiert durch

$$\alpha(q):=dx_1(q)\wedge dy_1(q)+dx_2(q)\wedge dy_2(q)+\cdots+dx_n(q)\wedge dy_n(q). \tag{10}$$

(a) Zeige, dass die von α definierte bilineare Form

$$\langle\cdot\,|\,\cdot\rangle : T_p[\mathbb{R}^{2n}] \times T_p[\mathbb{R}^{2n}] \to \mathbb{R}$$

nicht entartet ist und die Antisymmetrie-Eigenschaft erfüllt:

$$\forall w \in T_p[\mathbb{R}^{2n}] \langle v, w\rangle = 0 \Rightarrow v = 0, \quad \forall v, w \in T_p[\mathbb{R}^{2n}]\ \langle v|w\rangle = -\langle w|v\rangle. \tag{11}$$

(b) Zeige, dass $d\alpha = 0$.

(c) Zeige, dass

$$\underbrace{\alpha \wedge \alpha \wedge \cdots \wedge \alpha}_{n \text{ mal}} = n!\, dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.$$

Solution. Let M be a smooth manifold and let $\alpha \in T^2_0(M)$. Then α defines a bilinear form

$$\langle\cdot \mid \cdot\rangle : T_q[M] \times T_q[M] \to \mathbb{R},$$

which, in local coordinates $\phi = (U, x)$, can be written as

$$\left\langle \frac{\partial}{\partial x_i(q)} \,\middle|\, \frac{\partial}{\partial x_j(q)} \right\rangle = \alpha^{ij}(q), \tag{12}$$

where

$$\alpha(q) = \sum_{i,j=1}^{m} \alpha^{ij}(q)\, dx_i(q) \otimes dx_j(q). \tag{13}$$

In particular, for $q \in U$ the scalar product (12) can be expressed as

$$\langle v \mid w\rangle = \left\langle \Theta_{\phi,q}(v), (\alpha^{ij}(q))_{i,j}\, \Theta_{\phi,q}(w) \right\rangle_{\text{eucl}}, \qquad v, w \in T_q[M], \tag{14}$$

where $\langle\cdot \mid \cdot\rangle_{\text{eucl}} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ denotes the Euclidean inner product and $(\alpha^{ij}(q))_{i,j}$ is the $m \times m$ matrix with entries $\alpha^{ij}(q)$.

(a) Equation (10) yields

$$\alpha(q) = \sum_{j=1}^{n} \big(dx_j(q) \otimes dy_j(q) - dy_j(q) \otimes dx_j(q)\big). \tag{15}$$

In particular,

$$(\alpha^{ij})_{i,j} = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{pmatrix} =: M.$$

Therefore, for all $v, w \in T_q[\mathbb{R}^{2n}]$,

$$\langle v \mid w\rangle_{T_q[\mathbb{R}^{2n}]} = \langle \Theta_{\phi,q}(v), M\, \Theta_{\phi,q}(w)\rangle_{\text{eucl}}. \tag{16}$$

Suppose that $\langle v \mid w\rangle_{T_q[\mathbb{R}^{2n}]} = 0$ for all $w \in T_q[\mathbb{R}^{2n}]$. Since M is invertible, there exists $u \in \mathbb{R}^{2n}$ such that $Mu = \Theta_{\phi,q}(v)$. Taking $w = \Theta^{-1}_{\phi,q}(u)$, we obtain

$$\begin{aligned} 0 = \langle v \mid w\rangle_{T_q[\mathbb{R}^{2n}]} &= \big\langle \Theta_{\phi,q}(v), M\, \Theta_{\phi,q}(w) \big\rangle_{\text{eucl}} \\ &= \langle \Theta_{\phi,q}(v), Mu\rangle_{\text{eucl}} = \|\Theta_{\phi,q}(v)\|^2_{\text{eucl}}. \end{aligned} \tag{17}$$

Hence $\Theta_{\phi,q}(v) = 0$, and since $\Theta_{\phi,q}$ is an isomorphism, it follows that $v = 0$.

To verify antisymmetry, note that $M^T = -M$. Thus,

$$\begin{aligned}\langle v \mid w\rangle_{T_q\mathbb{R}^{2n}} &= \langle \Theta_{\phi,q}(v), M\,\Theta_{\phi,q}(w)\rangle_{\text{eucl}} \\ &= \langle M^T\Theta_{\phi,q}(v), \Theta_{\phi,q}(w)\rangle_{\text{eucl}} \\ &= -\langle M\Theta_{\phi,q}(v), \Theta_{\phi,q}(w)\rangle_{\text{eucl}} = -\langle w \mid v\rangle_{T_q\mathbb{R}^{2n}}.\end{aligned} \tag{18}$$

(c) Since $d(dx_j) = 0$ and, by the graded Leibniz rule,

$$d(dx_j \wedge dy_j) = (d\,dx_j) \wedge dy_j - dx_j \wedge (d\,dy_j) = 0,$$

and since d is linear, we conclude that $d\alpha = 0$.

(d) Let $\omega_i := dx_i \wedge dy_i \in \bigwedge^2(\mathbb{R}^{2n})$. Then

$$\omega_i \wedge \omega_j = (-1)^4\,\omega_j \wedge \omega_i = \omega_j \wedge \omega_i,$$

so ω_i and ω_j commute. Moreover, for $k \in \{1, \dots, n\}$,

$$\omega_k^2 = (dx_k \wedge dy_k) \wedge (dx_k \wedge dy_k) = -dx_k \wedge dx_k \wedge dy_k \wedge dy_k = 0. \tag{19}$$

Define $\alpha_j := \sum_{i=1}^{j} \omega_i$, so that $\alpha_n = \alpha$. We prove by induction that

$$(\alpha_j)^j = j!\,dx_1 \wedge dy_1 \wedge \cdots \wedge dx_j \wedge dy_j. \tag{20}$$

The base case follows from $\alpha_1 = \omega_1 = dx_1 \wedge dy_1$. Assume that (20) holds for some $j \in \{1, \dots, n-1\}$. Since ω_{j+1} commutes with α_j, we have

$$(\alpha_{j+1})^{j+1} = (\alpha_j + \omega_{j+1})^{j+1} = \sum_{i=0}^{j+1} \binom{j+1}{i} \alpha_j^{j+1-i} \wedge \omega_{j+1}^i. \tag{21}$$

By (19), $\omega_{j+1}^i = 0$ for all $i \geq 2$. Moreover, using the induction hypothesis,

$$(\alpha_j)^{j+1} = (\alpha_j)^j \wedge \alpha_j = j!\,\omega_1 \wedge \cdots \wedge \omega_j \wedge (\omega_1 + \cdots + \omega_j) = 0.$$

Substituting into (21), we obtain

$$\begin{aligned}(\alpha_{j+1})^{j+1} &= (j+1)(\alpha_j)^j \wedge \omega_{j+1} \\ &= (j+1)j!\,(\omega_1 \wedge \cdots \wedge \omega_j) \wedge \omega_{j+1} \\ &= (j+1)!\,dx_1 \wedge dy_1 \wedge \cdots \wedge dx_{j+1} \wedge dy_{j+1},\end{aligned} \tag{22}$$

which completes the induction.

Aufgabe 11.3 (2 + 3 + 3)

Sei E ein reeller Vektorraum, und sei

$$\beta = \{v_j : j \in \{1, \dots, n\}\}$$

eine Basis von E. Für $\alpha \in E^*$ sei die Kontraktion

$$i_\alpha : \bigwedge^p E \longrightarrow \bigwedge^{p-1} E$$

auf den Basisvektoren von $\bigwedge^p E$ definiert durch

$$i_\alpha(v_{i_1} \wedge \cdots \wedge v_{i_p}) := \sum_{j=1}^{p} (-1)^{j+1} \langle \alpha, v_{i_j} \rangle \, v_{i_1} \wedge \cdots \wedge v_{i_{j-1}} \wedge v_{i_{j+1}} \wedge \cdots \wedge v_{i_p}, \tag{23}$$

und linear fortgesetzt.

(a) Sei $\beta^* = \{v_j^* : j \in \{1, \ldots, n\}\}$ die duale Basis zu β. Zeige, dass für alle $p \leq n$ und $1 \leq i_1 < \cdots < i_p \leq n$ sowie für alle $j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_p\}$ gilt:

$$i_{v_j^*}(v_{i_1} \wedge \cdots \wedge v_{i_p}) = 0, \quad i_{v_j^*}(v_j \wedge v_{i_1} \wedge \cdots \wedge v_{i_p}) = v_{i_1} \wedge \cdots \wedge v_{i_p}. \tag{24}$$

(b) Zeige, dass für alle $\alpha \in E^*$, $v \in \bigwedge^p E$ und $w \in \bigwedge^q E$ gilt:

$$i_\alpha(v \wedge w) = (i_\alpha v) \wedge w + (-1)^p v \wedge (i_\alpha w). \tag{25}$$

Hinweis: Beachte, dass beide Seiten von (25) multilineare Abbildungen

$$E^* \times \bigwedge^p E \times \bigwedge^q E \longrightarrow \bigwedge^{p+q-1} E$$

sind. Daher genügt es, (25) für Basisvektoren zu beweisen.

(c) Sei M eine glatte Mannigfaltigkeit. Für ein Vektorfeld $X \in T_1^0[M]$ ist die Kontraktion mit X

$$i_X : \bigwedge^p [M] \longrightarrow \bigwedge^{p-1} [M]$$

definiert durch

$$(i_X \omega)(q) := i_{X(q)}(\omega(q)). \tag{26}$$

Zeige, dass für alle $\alpha \in \bigwedge^p [M]$ und $\beta \in \bigwedge^{\tilde{p}} [M]$ die gradierte Leibnizregel gilt:

$$i_X(\alpha \wedge \beta) = (i_X \alpha) \wedge \beta + (-1)^p \alpha \wedge (i_X \beta). \tag{27}$$

Solution.

(a) By definition, one has

$$\langle v_j^*, v_i \rangle = 0 \quad \text{for } i \neq j, \qquad \langle v_j^*, v_j \rangle = 1. \tag{28}$$

Therefore, for $j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_p\}$,

$$\begin{aligned} i_{v_j^*}(v_{i_1} \wedge \cdots \wedge v_{i_p}) &= \sum_{k=1}^{p} (-1)^{k+1} \langle v_j^*, v_{i_k} \rangle \, v_{i_1} \wedge \cdots \wedge v_{i_{k-1}} \wedge v_{i_{k+1}} \wedge \cdots \wedge v_{i_p} \\ &= 0. \end{aligned}$$

Suppose that $i_l < j < i_{l+1}$. Then

$$v_j \wedge v_{i_1} \wedge \cdots \wedge v_{i_p} = (-1)^l \, v_{i_1} \wedge \cdots \wedge v_{i_l} \wedge v_j \wedge v_{i_{l+1}} \wedge \cdots \wedge v_{i_p}.$$

Since $v_{i_1} \wedge \cdots \wedge v_{i_l} \wedge v_j \wedge v_{i_{l+1}} \wedge \cdots \wedge v_{i_p}$ is an element of a basis of $\bigwedge^{p+1} E$, we obtain

$$\begin{aligned} i_{v_j^*}(v_j \wedge v_{i_1} \wedge \cdots \wedge v_{i_p}) &= (-1)^l \, i_{v_j^*}(v_{i_1} \wedge \cdots \wedge v_{i_l} \wedge v_j \wedge v_{i_{l+1}} \wedge \cdots \wedge v_{i_p}) \\ &= (-1)^l (-1)^{l+2} \, v_{i_1} \wedge \cdots \wedge v_{i_p} = v_{i_1} \wedge \cdots \wedge v_{i_p}. \end{aligned} \tag{29}$$

(b) Since both the left-hand side and the right-hand side are multilinear functions on $E^* \times \bigwedge^p E \times \bigwedge^q E$, it suffices to verify the identity on basis elements. Let

$$v := v_{i_1} \wedge \cdots \wedge v_{i_p} \in \{v_{l_1} \wedge \cdots \wedge v_{l_p} : 1 \leq l_1 < \cdots < l_p \leq n\}$$

and

$$w := v_{j_1} \wedge \cdots \wedge v_{j_q} \in \{v_{l_1} \wedge \cdots \wedge v_{l_q} : 1 \leq l_1 < \cdots < l_q \leq n\}.$$

We distinguish the following cases.

Case 1. $\{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_q\} \neq \emptyset$.

In this case, $v \wedge w = 0$, and hence $i_\alpha(v \wedge w) = 0$. Moreover, if

$$|\{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_q\}| \geq 2,$$

then $(i_\alpha v) \wedge w = 0$ and $v \wedge (i_\alpha w) = 0$.

Assume now that

$$|\{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_q\}| = 1,$$

and let $i_k = j_l$. Since $(\{i_1, \ldots, i_p\} \setminus \{i_k\}) \cap \{j_1, \ldots, j_q\} = \emptyset$, we obtain

$$(i_\alpha v) \wedge w = (-1)^{k+1} \langle \alpha, v_{i_k} \rangle \, v_{i_1} \wedge \cdots \wedge v_{i_{k-1}} \wedge v_{i_{k+1}} \wedge \cdots \wedge v_{i_p} \wedge w. \tag{30}$$

Similarly,

$$v \wedge (i_\alpha w) = (-1)^{l+1} \langle \alpha, v_{i_l} \rangle \, v \wedge v_{j_1} \wedge \cdots \wedge v_{j_{l-1}} \wedge v_{j_{l+1}} \wedge \cdots \wedge v_{j_q}. \tag{31}$$

Note that

$$v_{i_1} \wedge \cdots \wedge v_{i_{k-1}} \wedge v_{i_{k+1}} \wedge \cdots \wedge v_{i_p} \wedge w = (-1)^{l-1+p-k} \, v \wedge v_{j_1} \wedge \cdots \wedge v_{j_{l-1}} \wedge v_{j_{l+1}} \wedge \cdots \wedge v_{j_q}. \tag{32}$$

Using (30)–(32) and the fact that $v_{i_k} = v_{i_l}$, we obtain

$$(i_\alpha v) \wedge w = (-1)^{p-1} \, v \wedge (i_\alpha w),$$

and therefore

$$(i_\alpha v) \wedge w + (-1)^p \, v \wedge (i_\alpha w) = 0.$$

Case 2. $\{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_q\} = \emptyset$.

Let $v_j^* \in \beta^*$. If $j \notin \{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\}$, then

$$i_{v_j^*}(v \wedge w) = 0, \qquad i_{v_j^*} v = 0, \qquad i_{v_j^*} w = 0.$$

Suppose that $j \in \{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\}$. If $j = i_k$, then

$$i_{v_j^*}(v \wedge w) = (-1)^{k+1} \, v_{i_1} \wedge \cdots \wedge v_{i_{k-1}} \wedge v_{i_{k+1}} \wedge \cdots \wedge v_{i_p} \wedge w,$$

and since $i_{v_j^*} w = 0$, we obtain

$$i_{v_j^*}(v \wedge w) = (i_{v_j^*} v) \wedge w = (i_{v_j^*} v) \wedge w + (-1)^p v \wedge (i_{v_j^*} w).$$

If $j = j_l$, then

$$i_{v_j^*}(v \wedge w) = (-1)^{p+l+1} \, v \wedge v_{j_1} \wedge \cdots \wedge v_{j_{l-1}} \wedge v_{j_{l+1}} \wedge \cdots \wedge v_{j_q},$$

and since $i_{v_j^*} v = 0$, we conclude

$$i_{v_j^*}(v \wedge w) = (-1)^p \, v \wedge (i_{v_j^*} w) = (i_{v_j^*} v) \wedge w + (-1)^p v \wedge (i_{v_j^*} w).$$

(c) Let $q \in M$. By definition and using part (b), we obtain

$$\begin{aligned}
(i_X(\alpha \wedge \beta))(q) &= (q, i_{X(q)}(\alpha(q) \wedge \beta(q))) \\
&= (q, (i_{X(q)}\alpha(q)) \wedge \beta(q) + (-1)^p \alpha(q) \wedge (i_{X(q)}\beta(q))) \\
&= ((i_X\alpha) \wedge \beta)(q) + (-1)^p (\alpha \wedge (i_X\beta))(q).
\end{aligned} \tag{33}$$