Technische Universität Braunschweig

10. Übungsblatt

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Deadline: 13.01.2026, 11.30 Uhr (vor der Übung).

Aufgabe 10.1 (6)

Sei M eine glatte Mannigfaltigkeit. Sei $\mathbb{X} \in T_1^0(M)$ ein Vektorfeld mit der Eigenschaft

$$[\mathbb{X}, \mathbb{Y}] = 0 \quad \text{für alle } \mathbb{Y} \in T_1^0(M).$$

Zeige, dass $\mathbb{X} = 0$ gilt.

Solution. Let $p \in M$ and let $\phi = (U, x) \in \mathcal{A}_M$ be a chart such that $p \in U$. In local coordinates, the vector field $\mathbb{X}$ can be written as

$$\mathbb{X}(q) = \Big(q, \sum_{k=1}^{n} X^k(q) \frac{\partial}{\partial x_k(q)}\Big), \qquad q \in U. \tag{1}$$

Fix $i, j \in \{1, \dots, n\}$. Denote by $\pi_j : \mathbb{R}^n \to \mathbb{R}$, $\pi_j(x) = x_j$, the projection onto the j-th coordinate and define $f_j \in C^\infty(M; \mathbb{R})$ by

$$f_j(q) = \pi_j \circ x(q), \qquad q \in V \subset U,$$

where $V \subset U$ is an open neighborhood of p.

Consider the vector field $\mathbb{Y} \in T_1^0[M]$ given by

$$\mathbb{Y}(q) = \Big(q, \frac{\partial}{\partial x_i(q)}\Big), \qquad q \in V.$$

Then, for every $q \in V$,

$$(L_{\mathbb{Y}} f_j)(q) = D_q[f_j]\left(\frac{\partial}{\partial x_i(q)}\right) = J_{\pi_j}(x(q))\, e_i = \delta_{ij}.$$

Hence $L_{\mathbb{Y}} f_j$ is constant on V, and therefore

$$L_{\mathbb{X}}(L_{\mathbb{Y}} f_j)(q) = 0, \qquad q \in V. \tag{2}$$

On the other hand, from (1) we obtain

$$(L_{\mathbb{X}} f_j)(q) = \sum_{k=1}^{n} X^k(q)\, D_q[f_j]\left(\frac{\partial}{\partial x_k(q)}\right) = X^j(q), \qquad q \in U. \tag{3}$$

Consequently, for $q \in V$,

$$L_{\mathbb{Y}}(L_{\mathbb{X}} f_j)(q) = \frac{\partial (X^j \circ x^{-1})}{\partial x_i}(x(q)). \tag{4}$$

By hypothesis, $L_{[\mathbb{X},\mathbb{Y}]} f_j = 0$. Using (2) and (4) we obtain

$$0 = L_{[\mathbb{X},\mathbb{Y}]} f_j(q) = L_{\mathbb{X}}(L_{\mathbb{Y}} f_j)(q) - L_{\mathbb{Y}}(L_{\mathbb{X}} f_j)(q) = -\frac{\partial (X^j \circ x^{-1})}{\partial x_i}(x(q)), \qquad q \in V.$$

Hence X^j is constant on V, say $X^j(q) = c_j \in \mathbb{R}$.

Fix j and define $\mathbb{Z} \in T_1^0[M]$ by

$$\mathbb{Z}(q) = \Big(q, c_j f_j(q) \frac{\partial}{\partial x_j(q)}\Big), \qquad q \in V.$$

Since $L_{\mathbb{X}} f_j(q) = c_j$ (see Eq. (3)) is constant , we have

$$L_{\mathbb{Z}}(L_{\mathbb{X}} f_j)(q) = 0, \qquad q \in V.$$

Moreover,

$$(L_{\mathbb{Z}} f_j)(q) = c_j f_j(q).$$

Therefore,

$$0 = L_{[\mathbb{X},\mathbb{Z}]} f_j(q) = L_{\mathbb{X}}(L_{\mathbb{Z}} f_j)(q) - L_{\mathbb{Z}}(L_{\mathbb{X}} f_j)(q) = c_j (L_{\mathbb{X}} f_j)(q) = c_j^2.$$

Thus $c_j = 0$. Since j was arbitrary,

$$\mathbb{X}(p) = (p, 0), \qquad \forall p \in M.$$

Aufgabe 10.2 [Jacobi-Identität] (6)

Sei M eine glatte Mannigfaltigkeit und seien $\mathbb{X}, \mathbb{Y}, \mathbb{Z} \in T_1^0(M)$. Zeige, dass

$$[\mathbb{X}, [\mathbb{Y}, \mathbb{Z}]] + [\mathbb{Y}, [\mathbb{Z}, \mathbb{X}]] + [\mathbb{Z}, [\mathbb{X}, \mathbb{Y}]] = 0$$

gilt.

Solution.

Using the definition one obtains

$$\begin{aligned} L_{[\mathbb{X},[\mathbb{Y},\mathbb{Z}]]} &= L_{\mathbb{X}} \circ L_{[\mathbb{Y},\mathbb{Z}]} - L_{[\mathbb{Y},\mathbb{Z}]} \circ L_{\mathbb{X}} \\ &= L_{\mathbb{X}} \circ \big(L_{\mathbb{Y}} \circ L_{\mathbb{Z}} - L_{\mathbb{Z}} \circ L_{\mathbb{Y}}\big) - \big(L_{\mathbb{Y}} \circ L_{\mathbb{Z}} - L_{\mathbb{Z}} \circ L_{\mathbb{Y}}\big) \circ L_{\mathbb{X}} \\ &= L_{\mathbb{X}} \circ L_{\mathbb{Y}} \circ L_{\mathbb{Z}} - L_{\mathbb{X}} \circ L_{\mathbb{Z}} \circ L_{\mathbb{Y}} - L_{\mathbb{Y}} \circ L_{\mathbb{Z}} \circ L_{\mathbb{X}} + L_{\mathbb{Z}} \circ L_{\mathbb{Y}} \circ L_{\mathbb{X}}. \end{aligned} \tag{5}$$

In the same way, one obtains

$$\begin{aligned} L_{[\mathbb{Y},[\mathbb{Z},\mathbb{X}]]} &= L_{\mathbb{Y}} \circ L_{\mathbb{Z}} \circ L_{\mathbb{X}} - L_{\mathbb{Y}} \circ L_{\mathbb{X}} \circ L_{\mathbb{Z}} \\ &\quad - L_{\mathbb{Z}} \circ L_{\mathbb{X}} \circ L_{\mathbb{Y}} + L_{\mathbb{X}} \circ L_{\mathbb{Z}} \circ L_{\mathbb{Y}}, \end{aligned} \tag{6}$$

and

$$\begin{aligned} L_{[\mathbb{Z},[\mathbb{X},\mathbb{Y}]]} &= L_{\mathbb{Z}} \circ L_{\mathbb{X}} \circ L_{\mathbb{Y}} - L_{\mathbb{Z}} \circ L_{\mathbb{Y}} \circ L_{\mathbb{X}} \\ &\quad - L_{\mathbb{X}} \circ L_{\mathbb{Y}} \circ L_{\mathbb{Z}} + L_{\mathbb{Y}} \circ L_{\mathbb{X}} \circ L_{\mathbb{Z}}. \end{aligned} \tag{7}$$

Adding equations (5), (6), and (7), we obtain

$$\begin{aligned} &L_{[\mathbb{X},[\mathbb{Y},\mathbb{Z}]]} + L_{[\mathbb{Y},[\mathbb{Z},\mathbb{X}]]} + L_{[\mathbb{Z},[\mathbb{X},\mathbb{Y}]]} \\ &\quad = L_{[\mathbb{X},[\mathbb{Y},\mathbb{Z}]]+[\mathbb{Y},[\mathbb{Z},\mathbb{X}]]+[\mathbb{Z},[\mathbb{X},\mathbb{Y}]]} = 0. \end{aligned}$$

Therefore,

$$[\mathbb{X}, [\mathbb{Y}, \mathbb{Z}]] + [\mathbb{Y}, [\mathbb{Z}, \mathbb{X}]] + [\mathbb{Z}, [\mathbb{X}, \mathbb{Y}]] = 0,$$

which proves the Jacobi identity and hence the result.

Aufhabe 10.3 [Lie-Algebra einer Lie-Gruppe] (3 + 3)

Eine Lie-Algebra $\mathcal{A}$ über $\mathbb{R}$ ist ein reeller Vektorraum zusammen mit einer bilinearen Abbildung

$$[\cdot,\cdot] : \mathcal{A} \times \mathcal{A} \to \mathcal{A},$$

genannt die Lie-Klammer, die folgende Eigenschaften erfüllt:

1. Bilinearität: Für alle $a \in \mathbb{R}$ und $X, Y, Z \in \mathcal{A}$ gilt
$$[aX + Y, Z] = a[X, Z] + [Y, Z], \qquad [X, aY + Z] = a[X, Y] + [X, Z].$$
2. Antisymmetrie:
$$[X, Y] = -[Y, X].$$
3. Jacobi-Identität:
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Sei G eine glatte Mannigfaltigkeit mit einer glatten Verknüpfung

$$\cdot : G \times G \to G,$$

sodass $(G, \cdot)$ eine Gruppe ist.

Für jedes $g \in G$ sei die Linkstranslation

$$L_g : G \to G, \qquad L_g(h) = g \cdot h.$$

Definiere

$$\mathfrak{g}_G = \{\mathbb{X} \in T^0_1(G) \mid (L_g)_*\mathbb{X} = \mathbb{X} \text{ für alle } g \in G\}.$$

(a) Zeige, dass $(\mathfrak{g}_G, [\cdot,\cdot])$ eine Lie-Algebra ist.

(b) Zeige, dass die Abbildung
$$r : \mathfrak{g}_G \to T_eG, \qquad r(\mathbb{X}) = \mathbb{X}(e),$$
wobei $e \in G$ das neutrale Element bezeichnet, ein linearer Isomorphismus ist.

Solution

(a) The set $T^0_1[M]$, endowed with the operations

$$(\mathbb{X} + \mathbb{Y})(q) = (q, X_q + Y_q), \qquad (s\mathbb{X})(q) = (q, sX_q),$$

is a vector space. Moreover, if $\mathbb{X}, \mathbb{Y} \in \mathfrak{g}_G$, then

$$\begin{aligned}
(L_g)_*(\mathbb{X} + \mathbb{Y}) &= D[L_g] \circ (\mathbb{X} + \mathbb{Y}) \circ (L_g)^{-1} \\
&= D[L_g] \circ \mathbb{X} \circ (L_g)^{-1} + D[L_g] \circ \mathbb{Y} \circ (L_g)^{-1} \\
&= (L_g)_*\mathbb{X} + (L_g)_*\mathbb{Y} = \mathbb{X} + \mathbb{Y}, \\
(L_g)_*(s\mathbb{X}) &= D[L_g] \circ (s\mathbb{X}) \circ (L_g)^{-1} = s(L_g)_*\mathbb{X} = s\mathbb{X}.
\end{aligned} \tag{8}$$

Hence, $\mathfrak{g}_G$ is a vector subspace of $T^0_1[M]$.

Next, we show that $\mathfrak{g}_G$ is closed under the Lie bracket. We use the general identity: for a diffeomorphism $\Phi : M \to N$, $X \in T^0_1[M]$, and $f \in C^\infty(N)$,

$$L_{\Phi_* X} f = L_X(f \circ \Phi) \circ \Phi^{-1}.$$

Let $f \in C^\infty(G; \mathbb{R})$ and $\mathbb{X}, \mathbb{Y} \in \mathfrak{g}_G$. Then

$$\begin{aligned} L_{(L_g)_*[\mathbb{X},\mathbb{Y}]} f &= L_{[\mathbb{X},\mathbb{Y}]}(f \circ L_g) \circ (L_g)^{-1} \\ &= L_{\mathbb{X}}\big(L_{\mathbb{Y}}(f \circ L_g)\big) \circ (L_g)^{-1} - L_{\mathbb{Y}}\big(L_{\mathbb{X}}(f \circ L_g)\big) \circ (L_g)^{-1} \\ &= L_{(L_g)_*\mathbb{X}} \circ L_{(L_g)_*\mathbb{Y}}(f) - L_{(L_g)_*\mathbb{Y}} \circ L_{(L_g)_*\mathbb{X}}(f) \\ &= L_{[(L_g)_*\mathbb{X},(L_g)_*\mathbb{Y}]} f = L_{[\mathbb{X},\mathbb{Y}]} f. \end{aligned}$$

Therefore,

$$(L_g)_*[\mathbb{X}, \mathbb{Y}] = [\mathbb{X}, \mathbb{Y}],$$

and hence $[\mathbb{X}, \mathbb{Y}] \in \mathfrak{g}_G$.

Finally, we verify that $[\cdot, \cdot]$ satisfies the axioms of a Lie bracket.

(a) Bilinearity. Let $\mathbb{X}, \mathbb{Y}, \mathbb{Z} \in T^0_1[G]$ and $r \in \mathbb{R}$. Then

$$\begin{aligned} L_{[r\mathbb{X}+\mathbb{Y},\mathbb{Z}]} &= L_{r\mathbb{X}+\mathbb{Y}} \circ L_{\mathbb{Z}} - L_{\mathbb{Z}} \circ L_{r\mathbb{X}+\mathbb{Y}} \\ &= r(L_{\mathbb{X}} \circ L_{\mathbb{Z}} - L_{\mathbb{Z}} \circ L_{\mathbb{X}}) + L_{\mathbb{Y}} \circ L_{\mathbb{Z}} - L_{\mathbb{Z}} \circ L_{\mathbb{Y}} \\ &= rL_{[\mathbb{X},\mathbb{Z}]} + L_{[\mathbb{Y},\mathbb{Z}]} = L_{r[\mathbb{X},\mathbb{Z}]+[\mathbb{Y},\mathbb{Z}]}. \end{aligned} \tag{9}$$

Hence,

$$[r\mathbb{X} + \mathbb{Y}, \mathbb{Z}] = r[\mathbb{X}, \mathbb{Z}] + [\mathbb{Y}, \mathbb{Z}].$$

Linearity in the second variable is analogous.

(b) Antisymmetry. For $\mathbb{X}, \mathbb{Y} \in T^0_1[G]$, a direct computation gives

$$L_{[\mathbb{X},\mathbb{Y}]} = -L_{[\mathbb{Y},\mathbb{X}]} = L_{-[\mathbb{X},\mathbb{Y}]},$$

which implies $[\mathbb{X}, \mathbb{Y}] = -[\mathbb{Y}, \mathbb{X}]$.

(c) Jacobi identity. This follows from Exercise 10.2.

(b) For $s \in \mathbb{R}$ and $\mathbb{X}, \mathbb{Y} \in \mathfrak{g}_G$,

$$\begin{aligned} r(s\mathbb{X} + \mathbb{Y}) &= \pi_2((s\mathbb{X} + \mathbb{Y})(e)) \\ &= s\pi_2(\mathbb{X}(e)) + \pi_2(\mathbb{Y}(e)) = sr(\mathbb{X}) + r(\mathbb{Y}), \end{aligned} \tag{10}$$

so r is linear.

We now show that r is surjective. Let $v \in T_e[G]$ and define

$$\mathbb{X}(g) := D[L_g](e, v) = (g, D_e[L_g]v).$$

The map $\mathbb{X}$ is smooth by the following general result: if $F : M \times N \to R$ is smooth and then $f_p : N \to R$, $f_p(q) = F(p, q)$ is smooth and

$$D[f_p](q, v) = D[F]((p, q), (0, v)).$$

Since the group multiplication $\cdot : G \times G \to G$ is smoot using the previous result we obtain

$$\mathbb{X} = D[\cdot] \circ Z$$

where $Z : G \to T[G \times G], Z(g) = ((g,e),(0,v))$ which is smooth and therefore $\mathbb{X}$ is smooth. Moreover, for all $g, g' \in G$,

$$\begin{aligned}((L_g)_*\mathbb{X})(g') &= D[L_g] \circ \mathbb{X} \circ (L_g)^{-1}(g') \\ &= D[L_g](g^{-1}g', D_e[L_{g^{-1}g'}]v) \\ &= (g', D_{g^{-1}g'}[L_g]D_e[L_{g^{-1}g'}]v) \\ &= (g', D_e[L_{g'}]v) = \mathbb{X}(g').\end{aligned} \tag{11}$$

Thus $\mathbb{X} \in \mathfrak{g}_G$. Since $L_e = I_G$ and $D_e[I_G] = I_{T_e[G]}$

$$r(\mathbb{X}) = \pi_2(e, D_e[L_e]v) = \pi_2(e,v) = v,$$

so r is surjective.

Finally, we show that r is injective. Suppose $r(\mathbb{X}) = 0$. Then, for any $g \in G$,

$$\begin{aligned}(e,0) = \mathbb{X}(e) &= ((L_{g^{-1}})_*\mathbb{X})(e) \\ &= D[L_{g^{-1}}] \circ \mathbb{X} \circ (L_{g^{-1}})^{-1}(e) \\ &= D[L_{g^{-1}}](g, X_g) = (e, D_g[L_{g^{-1}}]X_g).\end{aligned}$$

Hence $D_g[L_{g^{-1}}]X_g = 0$. Since $L_{g^{-1}}$ is a diffeomorphism, its differential is an isomorphism, and therefore $X_g = 0$. Thus $\mathbb{X}(g) = (g,0)$ for all $g \in G$.

Aufgabe 10.4 [Die orthogonale Gruppe] (3 + 3) Sei

$$O(n) := \{M \in \mathbb{R}^{n\times n} \mid M^T M = I_n\}, \qquad \mathrm{Sym}(n) := \{M \in \mathbb{R}^{n\times n} \mid M^T = M\}.$$

Betrachte die Abbildung

$$F : \mathbb{R}^{n\times n} \to \mathrm{Sym}(n), \qquad F(A) = AA^T - I_n.$$

(a) Zeige, dass

$$J_F(A)H = AH^T + HA^T,$$

und folgere, dass $O(n)$ eine glatte Untermannigfaltigkeit der Dimension $\frac{n(n-1)}{2}$ ist.

(b) Zeige, dass

$$T_A[O(n)] \cong \{H \in \mathbb{R}^{n\times n} \mid HA^T + AH^T = 0\}$$

gilt und dass

$$T[O(n)] \cong \{(A,H) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \mid HA^T + AH^T = 0\}.$$

(c) Verwende Übung 10.3(b) sowie den vorherigen Teil, um zu folgern, dass

$$\mathfrak{o}(n) := \mathfrak{g}_{O(n)} \cong \{H \in \mathbb{R}^{n\times n} \mid H^T + H = 0\}.$$

Solution.

(a) Let $A, H \in \mathbb{R}^{n\times n}$. By definition,

$$\begin{aligned} F(A+H) &= (A+H)(A+H)^T - I \\ &= AA^T - I + AH^T + HA^T + HH^T \\ &= F(A) + T_A(H) + HH^T, \end{aligned}$$

where

$$T_A : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}, \qquad T_A(H) = AH^T + HA^T,$$

which is a linear map. Since

$$\|HH^T\|_{\text{eukl}} \le \|H\|^2_{\text{eukl}}, \qquad \|H\|_{\text{eukl}} = \sqrt{\sum_{i,j} H_{ij}^2},$$

it follows that T_A is the differential of F at A, that is,

$$J_F(A)H = T_A(H) = AH^T + HA^T.$$

Now let $A \in O(n)$. For $M \in \text{Sym}(n)$, set

$$H := \frac{1}{2}MA.$$

Then

$$J_F(A)H = AH^T + HA^T = \frac{1}{2}AA^TM + \frac{1}{2}MAA^T = M.$$

Hence, for every $A \in O(n)$, the map

$$J_F(A) : \mathbb{R}^{n\times n} \to \text{Sym}(n)$$

is surjective.

Consider the linear isomorphism

$$L : \text{Sym}(n) \to \mathbb{R}^{n(n+1)/2}, \qquad L(M) = (M_{11}, M_{12}, M_{22}, \dots, M_{1n}, M_{nn}).$$

Define $\widetilde{F} = L \circ F$. Since L is linear, we have

$$J_{\widetilde{F}}(A) = L \circ J_F(A).$$

Therefore, $J_{\widetilde{F}}(A)$ is surjective for all $A \in O(n)$. Consequently,

$$O(n) = \widetilde{F}^{-1}(\{0\})$$

is a submanifold of $\mathbb{R}^{n\times n}$ of dimension

$$n^2 - \frac{n(n+1)}{2}.$$

Note that for $M, N \in O(n)$,

$$MN(MN)^T = MNN^TM^T = I,$$

hence $MN \in O(n)$. Thus, the multiplication map

$$\cdot : O(n) \times O(n) \to O(n)$$

is well defined. Since matrix multiplication

$$\cdot : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$$

is smooth, its restriction to $O(n) \times O(n)$ is also smooth.

(b) From the previous item, we obtain

$$T_A[O(n)] \cong \mathrm{Ker}(J_{\widetilde{F}}(A)) = \mathrm{Ker}(J_F(A)) = \{H \in \mathbb{R}^{n\times n} : HA^T + AH^T = 0\},$$

and therefore

$$T[O(n)] \cong \{(A,H) \in O(n) \times \mathbb{R}^{n\times n} : HA^T + AH^T = 0\}.$$

(c) Since $(O(n), \cdot)$ is a group and the multiplication map is smooth (Exercise 10.3), $O(n)$ is a Lie group. Hence, its Lie algebra $\mathfrak{g}_{O(n)}$ is isomorphic to

$$T_I[O(n)].$$

Using item (b), we obtain the desired characterization.