Technische Universität Braunschweig

9. Übungsblatt

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Deadline: 23.12.2025, 11.30 Uhr (vor der Übung).

Aufgabe 9.1 (6)

Sei

$$M := \{(a,b,c,d) \in \mathbb{R}^4 \mid ad - bc = 1\}.$$

Zeigen Sie, dass M eine Untermannigfaltigkeit von $\mathbb{R}^4$ ist und dass M parallelisierbar ist.

Solution. Consider the smooth map

$$F : \mathbb{R}^4 \to \mathbb{R}, \qquad F(a,b,c,d) = ad - bc - 1.$$

Then,

$$M = F^{-1}(0).$$

For any $(a,b,c,d) \in M$, the jacobian of F at (a,b,c,d) is given by

$$J_F(a,b,c,d) = (d, -c, -b, a),$$

which is nonzero at every point of M. Hence $J_F(a,b,c,d)$ is surjective, and then M is a smooth submanifold of $\mathbb{R}^4$ of dimension 3.

Moreover, for each $(a,b,c,d) \in M$,

$$\mathrm{Ker} J_F(a,b,c,d) = \{v \in \mathbb{R}^4 : \langle (d,-c,-b,a), v\rangle_{\mathrm{eukl}} = 0\}.$$

Therefore, the tangent bundle of M can be identified as

$$TM \cong \{((a,b,c,d), v) \in M \times \mathbb{R}^4 : \langle (d,-c,-b,a), v\rangle_{\mathrm{eukl}} = 0\}.$$

To show that M is parallelizable, it suffices to construct three smooth vector fields

$$X_1, X_2, X_3 \in T_1^0[M]$$

such that $\{V_1(q), V_2(q), V_3(q)\}$ is linearly independent for every $q \in M$ where $X_j(q) = (q, V_j(q))$.

Define $X_1, X_2, X_3 : M \to T[M]$ by

$$\begin{aligned}
X_1(a,b,c,d) &= ((a,b,c,d), (b,0,d,0)), \\
X_2(a,b,c,d) &= ((a,b,c,d), (0,a,0,c)), \\
X_3(a,b,c,d) &= ((a,b,c,d), (-a,b,-c,d)).
\end{aligned}$$

A direct computation shows that each of these vectors, lies in $\mathrm{Ker} J_F(a,b,c,d)$, hence $X_1, X_2, X_3 \in T_1^0[M]$.

We now verify pointwise linear independence. Suppose that

$$\lambda_1(b,0,d,0) + \lambda_2(0,a,0,c) + \lambda_3(-a,b,-c,d) = 0$$

for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. Considering the second and fourth components, we obtain

$$\lambda_2 a + \lambda_3 b = 0, \qquad \lambda_2 c + \lambda_3 d = 0,$$

or equivalently,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = 1,$$

it follows that $\lambda_2 = \lambda_3 = 0$. The remaining components then give

$$\lambda_1 b = \lambda_1 d = 0.$$

If $b = d = 0$, then $ad - bc = 0$, which contradicts the defining condition of M. Hence $\lambda_1 = 0$, and the vectors are linearly independent.

Therefore, $\{X_1, X_2, X_3\}$ defines a global frame for $T[M]$, and M is parallelizable.

Aufgabe 9.2 (3 + 3)

Seien M, N Mannigfaltigkeiten und $f : M \to N$ ein Diffeomorphismus.

(a) Für $p \in N$ definieren wir

$$f_p^* : T_p^*[N] \to T_{f^{-1}(p)}^*[M], \qquad f_p^*([g]_p^*) = [g \circ f]_{f^{-1}(p)}^*.$$

Zeigen Sie, dass f_p^* wohldefiniert ist und einen linearen Isomorphismus darstellt.

(b) Sei

$$f^* : T^*[N] \to T^*[M], \qquad f^*(p, [g]_p^*) = \big(f^{-1}(p), f_p^*([g]_p^*)\big).$$

Zeigen Sie, dass f^* ein Diffeomorphismus ist.

Solution.

(a) Consider $g, \tilde{g} \in \tilde{T}_p^*[N]$ such that $[g]_p^* = [\tilde{g}]_p^*$. Then there exists a chart $\phi = (U, x)$ with $p \in U$ such that

$$J_{g \circ x^{-1}}(x(p)) = J_{\tilde{g} \circ x^{-1}}(x(p)). \tag{1}$$

Let $\psi = (V, y)$ be a chart of M such that $f^{-1}(p) \in V$ and $V \subset f^{-1}(U)$. Then Eq. (1) implies

$$\begin{aligned} J_{g \circ f \circ y^{-1}}(y(f^{-1}(p))) &= J_{g \circ x^{-1} \circ x \circ f \circ y^{-1}}(y(f^{-1}(p))) \\ &= J_{g \circ x^{-1}}(x(p))\, J_{x \circ f \circ y^{-1}}(y(f^{-1}(p))) \\ &= J_{\tilde{g} \circ x^{-1}}(x(p))\, J_{x \circ f \circ y^{-1}}(y(f^{-1}(p))) \\ &= J_{\tilde{g} \circ f \circ y^{-1}}(y(f^{-1}(p))). \end{aligned} \tag{2}$$

Therefore,

$$[g \circ f]_{f^{-1}(p)}^* = [\tilde{g} \circ f]_{f^{-1}(p)}^*,$$

and hence f_p^* is well defined.

Now consider the linear isomorphisms

$$\Theta_{\phi,p}^* : T_p[N]^* \to \mathbb{R}^n, \qquad \Theta_{\phi,p}^*[g]_p^* = J_{g \circ x^{-1}}(x(p)), \tag{3}$$

and

$$\Theta_{\psi,f^{-1}(p)}^* : T_{f^{-1}(p)}[M]^* \to \mathbb{R}^n, \qquad \Theta_{\psi,f^{-1}(p)}^*[h]_{f^{-1}(p)}^* = J_{h \circ y^{-1}}(y(f^{-1}(p))). \tag{4}$$

Equation (2) implies

$$\Theta^*_{\psi,f^{-1}(p)} f^*_p([g]^*_p) = J_{g\circ f\circ y^{-1}}(y(f^{-1}(p))) = \Theta^*_{\phi,p}[g]^*_p\, J_{x\circ f\circ y^{-1}}(y(f^{-1}(p))). \tag{5}$$

Thus,

$$f^*_p = (\Theta^*_{\psi,f^{-1}(p)})^{-1} \circ R\big(J_{x\circ f\circ y^{-1}}(y(f^{-1}(p)))\big) \circ \Theta^*_{\phi,p}, \tag{6}$$

where

$$R\big(J_{x\circ f\circ y^{-1}}(y(f^{-1}(p)))\big) : \mathbb{R}^n \to \mathbb{R}^n$$

denotes right multiplication by the matrix $J_{x\circ f\circ y^{-1}}(y(f^{-1}(p)))$. Since f^*_p is a composition of isomorphisms, it is itself an isomorphism.

(b) The inverse of f^* is given by

$$(f^*)^{-1} : T^*[M] \to T^*[N], \qquad (f^*)^{-1}(q, [g]^*_q) = (f(q), (f^*_{f(q)})^{-1}([g]^*_q)).$$

We now show that f^* is smooth. Let $p_0 \in N$ and consider

$$(p_0, [g_0]^*_{p_0}) \in T^*[N], \qquad f^*(p_0, [g_0]^*_{p_0}) \in T^*[M].$$

Choose charts $\phi = (U, x)$ of N and $\psi = (V, y)$ of M such that $p_0 \in U$, $f^{-1}(p_0) \in V$, and $V \subset f^{-1}(U)$. Consider the induced charts

$$\Theta^*_\phi : T^*[U] \to x(U) \times \mathbb{R}^n, \qquad \Theta^*_\phi(p, [g]^*_p) = (x(p), \Theta^*_{\phi,p}[g]^*_p), \tag{7}$$

and

$$\Theta^*_\psi : T^*[V] \to y(V) \times \mathbb{R}^n, \qquad \Theta^*_\psi(q, [g]^*_q) = (y(q), \Theta^*_{\psi,q}[g]^*_q). \tag{8}$$

Define the map

$$F : x(f(V)) \times \mathbb{R}^n \to y(V) \times \mathbb{R}^n, \qquad F(a, v) = \big(y \circ f^{-1} \circ x^{-1}(a),\ v\, J_{x\circ f\circ y^{-1}}(y \circ f^{-1} \circ x^{-1}(a))\big),$$

which is smooth. For $(p, [g]^*_p) \in T^*[f(V)] \subset T^*[U]$, we have

$$\Theta^*_\psi \circ f^*(p, [g]^*_p) = \Theta^*_\psi(f^{-1}(p), f^*_p([g]^*_p)) = \big(y \circ f^{-1}(p), \Theta^*_{\psi,f^{-1}(p)} f^*_p([g]^*_p)\big). \tag{9}$$

On the other hand,

$$\begin{aligned} F \circ \Theta^*_\phi(p, [g]^*_p) &= F(x(p), \Theta^*_{\phi,p}[g]^*_p) \\ &= \big(y \circ f^{-1}(p),\ \Theta^*_{\phi,p}[g]^*_p J_{x\circ f\circ y^{-1}}(y \circ f^{-1}(p))\big) \\ &= \big(y \circ f^{-1}(p),\ \Theta^*_{\psi,f^{-1}(p)} f^*_p([g]^*_p)\big), \end{aligned} \tag{10}$$

where the last equality follows from Eq. (6). Equations (9) and (10) show that

$$\Theta^*_\psi \circ f^*|_{T^*[U]} \circ (\Theta^*_\phi)^{-1} = F,$$

and hence f^* is smooth. By a similar argument, $(f^*)^{-1}$ is also smooth.

Aufhabe 9.3 (2 + 4)

Seien $\mathbb{S}^2 = (S^2, \mathfrak{T}_{\text{rel}}, \mathcal{A})$ die 2-Sphäre mit $S^2 := \{q \in \mathbb{R}^3 : \|q\|_{\text{eukl}} = 1\}$ und $\phi = (U, x) \in \mathcal{A}$ eine Karte von $\mathbb{S}^2$, wobei

$$U = x^{-1}((0,\pi) \times (-\pi,\pi)) \quad \text{und} \quad x^{-1}(\vartheta,\alpha) := \begin{pmatrix} \sin(\vartheta)\cos(\alpha) \\ \sin(\vartheta)\sin(\alpha) \\ \cos(\vartheta) \end{pmatrix}.$$

(a) Berechnen Sie für jeden Punkt $q \in U$ die Basen

$$\left\{\frac{\partial}{\partial\vartheta(q)}, \frac{\partial}{\partial\alpha(q)}\right\} \subseteq T_q[\mathbb{S}^2] \quad \text{und} \quad \{d\vartheta(q), d\alpha(q)\} \subseteq T_q^*[\mathbb{S}^2]$$

(b) Seien $\mathbb{X}, \mathbb{Y} \in T_0^1[\mathbb{S}^2]$ zwei Vektorfelder mit $\mathbb{X}(q) = (q, X_q)$, $\mathbb{Y}(q) = (q, Y_q)$ und

$$X_q = X_q^{(\vartheta)} \frac{\partial}{\partial\vartheta(q)} + X_q^{(\alpha)} \frac{\partial}{\partial\alpha(q)}, \qquad Y_q = Y_q^{(\vartheta)} \frac{\partial}{\partial\vartheta(q)} + Y_q^{(\alpha)} \frac{\partial}{\partial\alpha(q)}.$$

Berechne die Wirkung von

$$[\mathbb{X}, \mathbb{Y}] := L_{\mathbb{X}} \circ L_{\mathbb{Y}} - L_{\mathbb{Y}} \circ L_{\mathbb{X}}$$

auf $C^\infty(\mathbb{S}^2; \mathbb{R})$ für jeden Punkt $q \in U$.

Solution.

(a) Consider $q \in U$ and write $x(q) = (\theta(q), \alpha(q))$. Choose $\delta > 0$ such that

$$\delta < \theta(q) < \pi - \delta, \qquad -\pi + \delta < \alpha(q) < \pi - \delta.$$

Define the curves

$$\gamma_1 : (-\delta,\delta) \to U, \qquad \gamma_1(t) := x^{-1}(\theta(q) + t, \alpha(q)),$$

and

$$\gamma_2 : (-\delta,\delta) \to U, \qquad \gamma_2(t) := x^{-1}(\theta(q), \alpha(q) + t).$$

Note that

$$\gamma_1(0) = \gamma_2(0) = x^{-1}(\theta(q), \alpha(q)) = q.$$

Hence, $\gamma_1, \gamma_2 \in \tilde{T}_q[\mathbb{S}^2]$, and therefore $[\gamma_1], [\gamma_2] \in T_q[\mathbb{S}^2]$. Moreover, for all $t \in (-\delta,\delta)$ one has

$$x \circ \gamma_1(t) = (\theta(q) + t, \alpha(q)),$$

which implies

$$\Theta_{\phi,q}[\gamma_1] = (x \circ \gamma_1)'(0) = (1,0).$$

In the same manner,

$$\Theta_{\phi,q}[\gamma_2] = (x \circ \gamma_2)'(0) = (0,1).$$

Therefore,

$$[\gamma_1] = \Theta_{\phi,q}^{-1}(e_1) = \frac{\partial}{\partial\theta(q)}, \qquad [\gamma_2] = \Theta_{\phi,q}^{-1}(e_2) = \frac{\partial}{\partial\alpha(q)}.$$

Let $p_1, p_2 : (0, \pi) \times (-\pi, \pi) \to \mathbb{R}$ denote the projections onto the first and second coordinates, respectively. Then, for all $q \in U$,

$$p_1 \circ x,\ p_2 \circ x \in \tilde{T}_q^*[M], \qquad [p_1 \circ x]_q^*,\ [p_2 \circ x]_q^* \in T_q^*[M].$$

Moreover,

$$\Theta_{\phi,q}^*([p_1 \circ x]_q^*) = J_{p_1}(x(q)) = (1, 0), \qquad \Theta_{\phi,q}^*([p_2 \circ x]_q^*) = J_{p_2}(x(q)) = (0, 1).$$

Therefore,

$$d\theta(q) = (\Theta_{\phi,q}^*)^{-1} e_1 = [p_1 \circ x]_q^*, \qquad d\alpha(q) = (\Theta_{\phi,q}^*)^{-1} e_2 = [p_2 \circ x]_q^*. \tag{11}$$

(b) Let $f \in C^\infty(\mathbb{S}^2, \mathbb{R})$. Then

$$\begin{aligned} (L_{\mathbb{X}} f)(q) &= D_q[f](X_q) \\ &= X_q^{(\theta)} D_q[f] \frac{\partial}{\partial \theta(q)} + X_q^{(\alpha)} D_q[f] \frac{\partial}{\partial \alpha(q)} \\ &= X_q^{(\theta)} \frac{\partial f \circ x^{-1}}{\partial x_1}(x(q)) + X_q^{(\alpha)} \frac{\partial f \circ x^{-1}}{\partial x_2}(x(q)). \end{aligned} \tag{12}$$

Here we used

$$D_q[f] \frac{\partial}{\partial \alpha(q)} = J_{f \circ x^{-1}} \circ \Theta_{\phi,q} \Theta_{\phi,q}^{-1}(e_2) = J_{f \circ x^{-1}}(x(q)) e_2 = \frac{\partial f \circ x^{-1}}{\partial x_2}(x(q)).$$

In a similar manner,

$$(L_{\mathbb{Y}} f)(q) = Y_q^{(\theta)} \frac{\partial f \circ x^{-1}}{\partial x_1}(x(q)) + Y_q^{(\alpha)} \frac{\partial f \circ x^{-1}}{\partial x_2}(x(q)). \tag{13}$$

Therefore,

$$\begin{aligned} L_{\mathbb{X}}(L_{\mathbb{Y}} f)(q) &= X_q^{(\theta)} \frac{\partial L_{\mathbb{Y}} f \circ x^{-1}}{\partial x_1}(x(q)) + X_q^{(\alpha)} \frac{\partial L_{\mathbb{Y}} f \circ x^{-1}}{\partial x_2}(x(q)) \\ &= X_q^{(\theta)} \Big(\frac{\partial Y_{x^{-1}}^{(\theta)}}{\partial x_1}(x(q)) \frac{\partial f \circ x^{-1}}{\partial x_1}(x(q)) + \frac{\partial^2 f \circ x^{-1}}{\partial x_1 \partial x_1}(x(q)) Y_q^{(\theta)} \Big) \\ &\quad + X_q^{(\theta)} \Big(\frac{\partial Y_{x^{-1}}^{(\alpha)}}{\partial x_1}(x(q)) \frac{\partial f \circ x^{-1}}{\partial x_2}(x(q)) + \frac{\partial^2 f \circ x^{-1}}{\partial x_2 \partial x_1}(x(q)) Y_q^{(\alpha)} \Big) \\ &\quad + X_q^{(\alpha)} \Big(\frac{\partial Y_{x^{-1}}^{(\theta)}}{\partial x_2}(x(q)) \frac{\partial f \circ x^{-1}}{\partial x_1}(x(q)) + \frac{\partial^2 f \circ x^{-1}}{\partial x_2 \partial x_1}(x(q)) Y_q^{(\theta)} \Big) \\ &\quad + X_q^{(\alpha)} \Big(\frac{\partial Y_{x^{-1}}^{(\alpha)}}{\partial x_2}(x(q)) \frac{\partial f \circ x^{-1}}{\partial x_2}(x(q)) + \frac{\partial^2 f \circ x^{-1}}{\partial x_2 \partial x_2}(x(q)) Y_q^{(\alpha)} \Big). \end{aligned} \tag{14}$$

Similarly,

$$
\begin{aligned}
L_{\mathbb{Y}}(L_{\mathbb{X}}f)(q) &= Y_q^{(\theta)} \frac{\partial L_{\mathbb{X}} f \circ x^{-1}}{\partial x_1}(x(q)) + Y_q^{(\alpha)} \frac{\partial L_{\mathbb{X}} f \circ x^{-1}}{\partial x_2}(x(q)) \\
&= Y_q^{(\theta)} \Big(\frac{\partial X_{x^{-1}}^{(\theta)}}{\partial x_1}(x(q)) \frac{\partial f \circ x^{-1}}{\partial x_1}(x(q)) + \frac{\partial^2 f \circ x^{-1}}{\partial x_1 \partial x_1}(x(q)) X_q^{(\theta)} \Big) \\
&\quad + Y_q^{(\theta)} \Big(\frac{\partial X_{x^{-1}}^{(\alpha)}}{\partial x_1}(x(q)) \frac{\partial f \circ x^{-1}}{\partial x_2}(x(q)) + \frac{\partial^2 f \circ x^{-1}}{\partial x_2 \partial x_1}(x(q)) X_q^{(\alpha)} \Big) \\
&\quad + Y_q^{(\alpha)} \Big(\frac{\partial X_{x^{-1}}^{(\theta)}}{\partial x_2}(x(q)) \frac{\partial f \circ x^{-1}}{\partial x_1}(x(q)) + \frac{\partial^2 f \circ x^{-1}}{\partial x_2 \partial x_1}(x(q)) X_q^{(\theta)} \Big) \\
&\quad + Y_q^{(\alpha)} \Big(\frac{\partial X_{x^{-1}}^{(\alpha)}}{\partial x_2}(x(q)) \frac{\partial f \circ x^{-1}}{\partial x_2}(x(q)) + \frac{\partial^2 f \circ x^{-1}}{\partial x_2 \partial x_2}(x(q)) X_q^{(\alpha)} \Big).
\end{aligned} \tag{15}
$$

Consequently,

$$
[\mathbb{X}, \mathbb{Y}] f(q) = [\mathbb{X}, \mathbb{Y}]_q^{(\theta)} \frac{\partial f \circ x^{-1}}{\partial x_1}(x(q)) + [\mathbb{X}, \mathbb{Y}]_q^{(\alpha)} \frac{\partial f \circ x^{-1}}{\partial x_2}(x(q)), \tag{16}
$$

where

$$
[\mathbb{X}, \mathbb{Y}]_q^{(\theta)} = \frac{\partial Y_{x^{-1}}^{(\theta)}}{\partial x_1} X_q^{(\theta)} + \frac{\partial Y_{x^{-1}}^{(\theta)}}{\partial x_2} X_q^{(\alpha)} - \frac{\partial X_{x^{-1}}^{(\theta)}}{\partial x_1} Y_q^{(\theta)} - \frac{\partial X_{x^{-1}}^{(\theta)}}{\partial x_2} Y_q^{(\alpha)}, \tag{17}
$$

and

$$
[\mathbb{X}, \mathbb{Y}]_q^{(\alpha)} = \frac{\partial Y_{x^{-1}}^{(\alpha)}}{\partial x_1} X_q^{(\theta)} + \frac{\partial Y_{x^{-1}}^{(\alpha)}}{\partial x_2} X_q^{(\alpha)} - \frac{\partial X_{x^{-1}}^{(\alpha)}}{\partial x_1} Y_q^{(\theta)} - \frac{\partial X_{x^{-1}}^{(\alpha)}}{\partial x_2} Y_q^{(\alpha)}. \tag{18}
$$

Aufgabe 9.4 (2 + 2 + 2)

Sei M eine n-dimensionale Mannigfaltigkeit mit Atlas $\mathcal{A}$.

(a) Sei $\phi = (U, x) \in \mathcal{A}$. Für jedes $q \in U$ definieren wir

$$
\tilde{\Theta}_{\phi,q} : (T_q[M])^* \to \mathbb{R}^n, \qquad \tilde{\Theta}_{\phi,q}(\varphi) = \big(\varphi(\Theta_{\phi,q}^{-1}(e_1)), \ldots, \varphi(\Theta_{\phi,q}^{-1}(e_n))\big),
$$

wobei $\{e_1, \ldots, e_n\}$ die Standardbasis von $\mathbb{R}^n$ bezeichnet. Zeigen Sie, dass $\tilde{\Theta}_{\phi,q}$ ein linearer Isomorphismus ist.

(b) Für jedes $\phi = (U, x) \in \mathcal{A}$ betrachten wir die Menge

$$
T[U]^* := \bigcup_{q \in U} \{q\} \times (T_q[M])^*
$$

sowie die Abbildung

$$
\tilde{\Theta}_\phi : T[U]^* \to x(U) \times \mathbb{R}^n, \qquad \tilde{\Theta}_\phi(q, \varphi) = \big(x(q), \tilde{\Theta}_{\phi,q}(\varphi)\big).
$$

Zeigen Sie, dass die Menge

$$
\mathfrak{T}_{T[M]^*} := \Big\{ W \subset T[M]^* \;\Big|\; \tilde{\Theta}_\phi\big(W \cap T[U]^*\big) \subset \mathbb{R}^{2n} \text{ offen ist für alle } \phi = (U, x) \in \mathcal{A} \Big\}
$$

eine Topologie auf

$$
T[M]^* := \bigcup_{q \in M} \{q\} \times (T_q[M])^*
$$

definiert und dass

$$\mathcal{A}_{T[M]^*} := \left\{ (T[U]^*, \tilde{\Theta}_\phi) : \phi = (U,x) \in \mathcal{A} \right\}$$

ein Atlas von $T[M]^*$ ist.

(c) Betrachten Sie die Abbildung

$$\Phi : T^*[M] \to T[M]^*, \qquad \Phi(q, [f]_q^*) = \big(q, \Phi_q([f]_q^*)\big),$$

wobei $\Phi_q : T_q^*[M] \to (T_q[M])^*$ durch $\Phi_q([f]_q^*)([\gamma]) = \langle [f]_q^*, [\gamma] \rangle$ definiert ist. Zeigen Sie, dass Φ ein Diffeomorphismus ist.

Solution.

(a) It is clear that $\tilde{\Theta}_{\phi,q}$ is linear. Since

$$\dim\big((T_q[M])^*\big) = \dim(T_q[M]) = n,$$

to show that $\tilde{\Theta}_{\phi,q}$ is an isomorphism it suffices to prove that it is injective. Let $\varphi \in (T_q[M])^*$ be such that $\tilde{\Theta}_{\phi,q}(\varphi) = 0$. Then, for all $j \in \{1, \ldots, n\}$,

$$\varphi(\Theta_{\phi,q}^{-1}(e_j)) = 0. \tag{19}$$

Since

$$\{\Theta_{\phi,q}^{-1}(e_j)\}_{j=1,\ldots,n} = \left\{ \frac{\partial}{\partial x_j(q)} \right\}_{j=1,\ldots,n}$$

is a basis of $T_q[M]$, it follows that $\varphi([\gamma]) = 0$ for all $[\gamma] \in T_q[M]$, and therefore $\varphi = 0$. Hence, $\tilde{\Theta}_{\phi,q}$ is injective and thus an isomorphism.

(b) First, note that $x : U \to x(U)$ is a bijection (in fact, a homeomorphism), and for all $q \in U$ the map $\tilde{\Theta}_{\phi,q}$ is bijective (since it is an isomorphism). Therefore, the map $\tilde{\Theta}_\phi$ is bijective for every $\phi = (U,x) \in \mathcal{A}_M$. Its inverse is given by

$$\tilde{\Theta}_\phi^{-1} : x(U) \times \mathbb{R}^n \to T[U]^*, \qquad \tilde{\Theta}_\phi^{-1}(a,v) = \big(x^{-1}(a), (\tilde{\Theta}_{\phi,x^{-1}(a)})^{-1} v\big). \tag{20}$$

We now show that $\mathfrak{T}_{T[M]^*}$ is a topology. If $W = \emptyset$, then for all $\phi \in \mathcal{A}_M$,

$$\tilde{\Theta}_\phi(\emptyset \cap T[U]^*) = \tilde{\Theta}_\phi(\emptyset) = \emptyset \in \mathfrak{T}_{\mathbb{R}^{2n}},$$

and hence $\emptyset \in \mathfrak{T}_{T[M]^*}$. If $W = T[M]^*$, then for all $\phi = (U,x)$,

$$\tilde{\Theta}_\phi(T[M]^* \cap T[U]^*) = \tilde{\Theta}_\phi(T[U]^*) = x(U) \times \mathbb{R}^n \in \mathfrak{T}_{\mathbb{R}^{2n}}.$$

Let $\{W_\alpha\}_{\alpha \in I} \subset \mathfrak{T}_{T[M]^*}$. For any $\phi \in \mathcal{A}_M$, using that $\tilde{\Theta}_\phi$ is bijective,

$$\tilde{\Theta}_\phi\left(\bigcup_{\alpha \in I} W_\alpha \cap T[U]^* \right) = \bigcup_{\alpha \in I} \tilde{\Theta}_\phi(W_\alpha \cap T[U]^*) \in \mathfrak{T}_{\mathbb{R}^{2n}}. \tag{21}$$

Thus, $\bigcup_{\alpha \in I} W_\alpha \in \mathfrak{T}_{T[M]^*}$.

Now assume $W_j \in \mathfrak{T}_{T[M]^*}$ for $j \in \{1, \ldots, n\}$. Then,

$$\tilde{\Theta}_\phi \left(\bigcap_{j=1}^{n} W_j \cap T[U]^* \right) = \bigcap_{j=1}^{n} \tilde{\Theta}_\phi(W_j \cap T[U]^*) \in \mathfrak{T}_{\mathbb{R}^{2n}}. \tag{22}$$

Hence, $\mathfrak{T}_{T[M]^*}$ is a topology.

We now verify that $\mathcal{A}_{T[M]^*}$ is an atlas. Let $\phi = (U, x)$ and $\psi = (V, y)$ be elements of $\mathcal{A}_M$ with $U \cap V \neq \emptyset$, and let $p \in U \cap V$. For $i \in \{1, \ldots, n\}$, let $\Theta_{\psi,p}^{-1}(e_i) = [\gamma]$ for some curve $\gamma : (-a, a) \to U \cap V$ such that $\gamma(0) = p$ and $(y \circ \gamma)'(0) = e_i$. By definition and the chain rule,

$$\Theta_{\phi,p}\Theta_{\psi,p}^{-1}(e_i) = \Theta_{\phi,p}[\gamma] = (x \circ \gamma)'(0) = (x \circ y^{-1} \circ y \circ \gamma)'(0) = J_{x \circ y^{-1}}(y(p))e_i. \tag{23}$$

Equation (23) implies that, for all $p \in U \cap V$,

$$\Theta_{\psi,p}^{-1}(e_i) = \sum_{j=1}^{n} J_{x \circ y^{-1}}(y(p))_{j,i}\, \Theta_{\phi,p}^{-1}(e_j). \tag{24}$$

Let $v \in \mathbb{R}^n$ and define $\varphi = \tilde{\Theta}_{\phi,p}^{-1}(v)$, so that $\varphi(\Theta_{\phi,p}^{-1}(e_i)) = v_i$. Using the definition and Eq. (24), we obtain

$$\begin{aligned} \tilde{\Theta}_{\psi,p}\tilde{\Theta}_{\phi,p}^{-1}(v) &= \tilde{\Theta}_{\psi,p}(\varphi) = (\varphi(\Theta_{\psi,p}^{-1}(e_1)), \ldots, \varphi(\Theta_{\psi,p}^{-1}(e_n))) \\ &= \sum_{j=1}^{n} (J_{x \circ y^{-1}}(y(p))_{j,1}, \ldots, J_{x \circ y^{-1}}(y(p))_{j,n}) v_j = (J_{x \circ y^{-1}}(y(p))^T v)^T. \end{aligned} \tag{25}$$

Equations (25) and (20) show that the transition map

$$\tilde{\Theta}_\psi \circ \tilde{\Theta}_\phi^{-1} : x(U \cap V) \times \mathbb{R}^n \to y(U \cap V) \times \mathbb{R}^n$$

is given by

$$\tilde{\Theta}_\psi \circ \tilde{\Theta}_\phi^{-1}(a, v) = \big(y \circ x^{-1}(a), (J_{x \circ y^{-1}}(y \circ x^{-1}(a))^T v)^T\big), \tag{26}$$

which is a diffeomorphism. Hence, the change of charts is smooth.

It remains to show that $\tilde{\Theta}_\phi$ is a homeomorphism. By definition, it is open: if $W' \subset T[U]^*$ is open, then $W' = W \cap T[U]^*$ for some $W \in \mathfrak{T}_{T[M]^*}$, then from the definition of $\mathfrak{T}_{T[M]^*}$ it follows that

$$\tilde{\Theta}_\phi(W') = \tilde{\Theta}_\phi(W \cap T[U]^*) \in \mathfrak{T}_{\mathbb{R}^{2n}}.$$

To prove continuity, let $O \in \mathfrak{T}_{\mathbb{R}^{2n}}$ with $O \subset x(U) \times \mathbb{R}^n$. We must show that $\tilde{\Theta}_\phi^{-1}(O) \in \mathfrak{T}_{T[M]^*}$. For $\psi = (V, y) \in \mathcal{A}_M$,

$$\tilde{\Theta}_\psi\big(T[V]^* \cap \tilde{\Theta}_\phi^{-1}(O)\big) = \tilde{\Theta}_\psi \circ \tilde{\Theta}_\phi^{-1}\big(O \cap (x(U \cap V) \times \mathbb{R}^n)\big). \tag{27}$$

If $U \cap V = \emptyset$, the right-hand side is empty. If $U \cap V \neq \emptyset$, Eq. (26) shows that $\tilde{\Theta}_\psi \circ \tilde{\Theta}_\phi^{-1}$ is a homeomorphism, and hence the image is open in $\mathbb{R}^{2n}$. In either case, Eq. (27) implies that $\tilde{\Theta}_\phi^{-1}(O) \in \mathfrak{T}_{T[M]^*}$. Therefore, $\tilde{\Theta}_\phi$ is continuous, and since it is an open, continuous bijection, it is a homeomorphism.

(c) Let $p \in M$ and choose $\phi = (U, x) \in \mathcal{A}_M$ such that $p \in U$. Consider the charts

$$\Theta^*_\phi : T^*[U] \to x(U) \times \mathbb{R}^n, \qquad \Theta^*_\phi(q, [f]^*_q) = (x(q), J_{f \circ x^{-1}}(x(q))), \tag{28}$$

and

$$\tilde{\Theta}_\phi : T[U]^* \to x(U) \times \mathbb{R}^n, \qquad \tilde{\Theta}_\phi(q, \varphi) = (x(q), \tilde{\Theta}_{\phi,q}(\varphi)). \tag{29}$$

For any $(q, [f]^*_q) \in T^*[U]$,

$$\begin{aligned} \tilde{\Theta}_\phi \circ \Phi(q, [f]^*_q) &= \tilde{\Theta}_\phi(q, \Phi_q([f]^*_q)) \\ &= (x(q), \tilde{\Theta}_{\phi,q}\Phi_q([f]^*_q)) \\ &= (x(q), (\langle [f]^*_q, \Theta^{-1}_{\phi,q}(e_1)\rangle, \ldots, \langle [f]^*_q, \Theta^{-1}_{\phi,q}(e_n)\rangle)) \\ &= (x(q), J_{f \circ x^{-1}}(x(q))) = \Theta^*_\phi(q, [f]^*_q). \end{aligned} \tag{30}$$

Thus,

$$\tilde{\Theta}_\phi \circ \Phi| \circ (\Theta^*_\phi)^{-1} = \mathrm{id}_{x(U) \times \mathbb{R}^n}, \qquad \Theta^*_\phi \circ \Phi^{-1}| \circ \tilde{\Theta}^{-1}_\phi = \mathrm{id}_{x(U) \times \mathbb{R}^n},$$

and both maps are smooth.