Technische Universität Braunschweig

8. Übungsblatt

Upload: 09.12.2025
Deadline: 16.12.2025, 11.30 Uhr (vor der Übung).

Aufgabe 8.1 (3 + 3 + 2)

Sei $M \subset \mathbb{R}^n$ eine m-dimensionale Untermannigfaltigkeit.

(a) Zeigen Sie, dass der Raum

$$T_p^{ext}[M] := \{\gamma'(0) \in \mathbb{R}^n : \gamma \in \tilde{T}_p[M]\}$$

ein m-dimensionaler Untervektorraum von $\mathbb{R}^n$ ist, und dass die Abbildung

$$\begin{aligned} T_p[M] &\to T_p^{ext}[M], \\ [\gamma] &\mapsto \gamma'(0) \end{aligned}$$

wohldefiniert ist und ein Isomorphismus darstellt.

(b) Zeigen Sie, dass die Menge

$$T^{ext}[M] := \{(p, v) \in M \times \mathbb{R}^n : v \in T_p^{ext}[M]\} \subset \mathbb{R}^n \times \mathbb{R}^n$$

eine Untermannigfaltigkeit der Dimension $2m$ ist.

(c) Zeigen Sie, dass die Abbildung

$$\Phi : T[M] \to T^{ext}[M], \quad \Phi(p, [\gamma]) = (p, \gamma'(0)),$$

ein Diffeomorphismus ist.

Solution.

(a) Since M is a submanifold of $\mathbb{R}^n$, there exists a diffeomorphism

$$\phi : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^n$$

such that

$$\phi(U \cap M) = (\mathbb{R}^m \times \{0\}) \cap V. \tag{1}$$

We prove that for all $p \in U \cap M$,

$$T_p^{\text{ext}}[M] = J_{\phi^{-1}}(\phi(p))(\mathbb{R}^m \times \{0\}). \tag{2}$$

Let $v \in T_p^{\text{ext}}[M]$. Then there exists a curve $\gamma \in C^1((a, b), M)$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. Since $\gamma(t) \in M$, Equation (1) implies

$$\phi \circ \gamma(t) = (c_1(t), \ldots, c_m(t), 0, \ldots, 0).$$

Therefore,

$$(c_1'(0), \ldots, c_m'(0), 0, \ldots, 0) = (\phi \circ \gamma)'(0) = J_\phi(p)\gamma'(0),$$

and hence

$$v = \gamma'(0) = J_{\phi^{-1}}(\phi(p))(c_1'(0), \ldots, c_m'(0), 0, \ldots, 0) \in J_{\phi^{-1}}(\phi(p))(\mathbb{R}^m \times \{0\}).$$

Conversely, let $v \in J_{\phi^{-1}}(\phi(p))(\mathbb{R}^m \times \{0\})$. Then there exists $w = (w_1, \ldots, w_m, 0, \ldots, 0)$ such that $v = J_{\phi^{-1}}(\phi(p))w$. Define the curve

$$\gamma(t) := \phi^{-1}(tw + \phi(p)).$$

Equation (1) implies that $\gamma(t) \in U \cap M$. Moreover, $\gamma(0) = p$ and

$$\gamma'(0) = J_{\phi^{-1}}(\phi(p))w = v.$$

This proves (2). In particular, $T_p^{\mathrm{ext}}[M]$ is an m-dimensional vector space.

For the second part, consider the chart $\psi = (x, U \cap M)$, where $x : U \cap M \to \mathbb{R}^m$ satisfies

$$\phi|_{U \cap M} = (x, 0)$$

Then

$$\phi \circ \gamma = (x \circ \gamma, 0),$$

and in particular,

$$(\phi \circ \gamma)'(0) = ((x \circ \gamma)'(0), 0).$$

If $\gamma \sim \tilde{\gamma}$, then $(x \circ \gamma)'(0) = (x \circ \tilde{\gamma})'(0)$, which implies

$$(\phi \circ \gamma)'(0) = (\phi \circ \tilde{\gamma})'(0).$$

Consequently,

$$\gamma'(0) = (\phi^{-1} \circ \phi \circ \gamma)'(0) = J_{\phi^{-1}}(\phi(p))(\phi \circ \gamma)'(0) = \tilde{\gamma}'(0).$$

This shows that the map is well defined.

Moreover, consider the linear isomorphism

$$\Theta_{\psi,p} : T_p[M] \to \mathbb{R}^m, \qquad \Theta_{\psi,p}[\gamma] = (x \circ \gamma)'(0).$$

Then

$$L_p[\gamma] = J_{\phi^{-1}}(\phi(p))(\Theta_{\psi,p}[\gamma], 0).$$

This above implies that L_p is injective. Since both spaces have the same dimension, $L_p : T_p[M] \to T_p^{\mathrm{ext}}[M]$ is a linear isomorphism.

(b) Let $p \in M$ and let

$$\phi_p : U_p \subset \mathbb{R}^n \to V_p \subset \mathbb{R}^n$$

be a diffeomorphism as in (1), with $p \in U_p$. Note that for every $q \in U_p$ and every $v = \gamma'(0) \in T_q^{\mathrm{ext}}[M]$, one has

$$J_{\phi_p}(q)\gamma'(0) = (\phi_p \circ \gamma)'(0) \in \mathbb{R}^m \times \{0\} \subset \mathbb{R}^n.$$

Consider the diffeomorphism

$$\Psi_p : U_p \times \mathbb{R}^n \to V_p \times \mathbb{R}^n, \qquad \Psi_p(q, v) = \big(\phi_p(q), J_{\phi_p}(q)v\big).$$

Then
$$\Psi_p\big((U_p \times \mathbb{R}^n) \cap T^{\text{ext}}[M]\big) = \big((\mathbb{R}^m \times \{0\}) \cap V_p\big) \times (\mathbb{R}^m \times \{0\}).$$
This shows that $T^{\text{ext}}[M]$ is a submanifold of $\mathbb{R}^n \times \mathbb{R}^n$.

Moreover, an atlas for $T^{\text{ext}}[M]$ is given by
$$\mathcal{A}_{T^{\text{ext}}[M]} := \big\{\eta_p = \big((U_p \times \mathbb{R}^n) \cap T^{\text{ext}}[M],\, y_p\big) :\ p \in M\big\}\,, \tag{3}$$
where the coordinate map y_p is defined by
$$y_p(q,v) = \big(\pi_m(\phi_p(q)), \pi_m(J_{\phi_p}(q)v)\big).$$
Here
$$\pi_m : \mathbb{R}^n \to \mathbb{R}^m, \qquad \pi_m(x_1, \ldots, x_n) = (x_1, \ldots, x_m).$$

(c) Let $(p, [\gamma]) \in T[M]$ and let
$$\phi_p : U_p \subset \mathbb{R}^n \to V_p \subset \mathbb{R}^n$$
be as in (1), with $p \in U_p$. Consider the chart $\psi_p = (U_p \cap M, x_p)$, where $\phi_p|_{U_p \cap M} = (x_p, 0)$. Let
$$\Theta_{\psi_p} : T[U_p \cap M] \to x_p(U_p \cap M) \times \mathbb{R}^m, \qquad \Theta_{\psi_p}(q, [\gamma]) = \big(x_p(q), (x_p \circ \gamma)'(0)\big)$$
be the induced chart on $T[M]$, and let $\eta_p = ((U_p \times \mathbb{R}^n) \cap T^{\text{ext}}[M], y_p)$ be the chart of $T^{\text{ext}}[M]$ defined in (3).

For $(q, [\gamma]) \in T[U_p \cap M]$, one has
$$\begin{aligned} y_p \circ \Phi(q, [\gamma]) &= y_p(q, \gamma'(0)) \\ &= \big(\pi_m(\phi_p(q)), \pi_m((\phi_p \circ \gamma)'(0))\big) \\ &= \big(x_p(q), (x_p \circ \gamma)'(0)\big) \\ &= \Theta_{\psi_p}(q, [\gamma]). \end{aligned}$$

Therefore,
$$y_p \circ \Phi| \circ \Theta_{\psi_p}^{-1} = \mathrm{id}_{x_p(U_p \cap M) \times \mathbb{R}^m},$$
which is smooth.

The inverse of Φ is given by
$$\Phi^{-1} : T^{\text{ext}}[M] \to T[M], \qquad \Phi^{-1}(q, v) = (q, L_q^{-1}(v)).$$

Moreover,
$$\Theta_{\psi_p} \circ (\Phi|)^{-1} \circ y_p^{-1} = \mathrm{id}_{x_p(U_p \cap M) \times \mathbb{R}^m}.$$
Hence, $\Phi : T[M] \to T^{\text{ext}}[M]$ is a diffeomorphism.

Aufgabe 8.2 (8)

Sei $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ eine glatte Funktion, sodass für alle $p \in F^{-1}(0)$ die Jacobimatrix $J_F(p)$ surjektiv ist. Bezeichnen wir $M = F^{-1}(0)$, wodurch M eine Untermannigfaltigkeit wird. Zeigen Sie, dass
$$T^{ext}[M] = \{(p, v) \in M \times \mathbb{R}^{n+m} : v \in \mathrm{Ker}(J_F(p))\}.$$

Solution. Let $p \in M$. We prove that

$$\mathrm{Ker}(J_F(p)) = T_p^{\mathrm{ext}}[M], \tag{4}$$

which together with the definiton of $T^{ext}[M]$ will imply the desired.

Since $F^{-1}(\{0\})$ is a submanifold, for $p \in F^{-1}(\{0\})$ there exists a diffeomorphism

$$\phi : U \subset \mathbb{R}^{n+m} \to V \subset \mathbb{R}^{n+m}$$

such that $\phi(p) = 0$ and

$$\phi(U \cap M) = (\mathbb{R}^n \times \{0\}) \cap B_r(0).$$

Note that for all $x \in \mathbb{R}^n$ with $|x| < r$, one has $\phi^{-1}(x, 0) \in M = F^{-1}(0)$, and thus

$$F \circ \phi^{-1} \circ s(x) = 0, \ |x| < r,$$

with $s : \mathbb{R}^n \to \mathbb{R}^{n+m}, \ s(x) = (x, 0)$. Taking the derivative at 0 and using the chain rule we obtian

$$0 = J_F(p)\, J_{\phi^{-1}}(0) J_s(\mathbb{R}^n) = J_F(p)\, J_{\phi^{-1}}(0)(\mathbb{R}^n \times \{0\}),$$

which implies

$$J_{\phi^{-1}}(0)(\mathbb{R}^n \times \{0\}) \subset \mathrm{Ker}(J_F(p)).$$

Since

$$\dim(\mathrm{Ker}(J_F(p))) = n = \dim(J_{\phi^{-1}}(0)(\mathbb{R}^n \times \{0\})),$$

we conclude that

$$J_{\phi^{-1}}(0)(\mathbb{R}^n \times \{0\}) = \mathrm{Ker}(J_F(p)).$$

Finally, Eq. (2) implies

$$T_p^{\mathrm{ext}}[M] = J_{\phi^{-1}}(0)(\mathbb{R}^n \times \{0\}),$$

which completes the proof.

Aufgabe 8.3 (8)

Sei $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ die n-dimensionale Sphäre.

(a) Zeigen Sie, dass

$$T_p^{ext}[\mathbb{S}^n] = \{v \in \mathbb{R}^{n+1} : p \cdot v = 0\}$$

gilt und dass

$$T^{ext}[\mathbb{S}^n] = \{(p, v) \in \mathbb{S}^n \times \mathbb{R}^{n+1} : p \cdot v = 0\}.$$

(b) Zeigen Sie, dass für ungerades n eine glatte Abbildung

$$X : \mathbb{S}^n \to T[\mathbb{S}^n]$$

existiert, sodass

$$\pi(X(p)) = p,$$

wobei $\pi : T[\mathbb{S}^n] \to \mathbb{S}^n, \quad \pi(p, [\gamma]) = p$ die kanonische Projektion des Tangentialbündels ist, und sodass $X(p) \neq 0$ für alle $p \in \mathbb{S}^n$ gilt.

(c) Zeigen Sie, dass es einen Diffeomorphismus

$$\Phi : \mathbb{S}^1 \times \mathbb{R} \to T[\mathbb{S}^1]$$

gibt, sodass $\pi(\Phi(p, t)) = p$ und $\Phi|_{\{p\} \times \mathbb{R}} : \{p\} \times \mathbb{R} \to T_p\mathbb{S}^1$ ein linearer Isomorphismus ist.

Solution

(a) Consider the map

$$F : \mathbb{R}^{n+1} \to \mathbb{R}, \qquad F(x) = \|x\|^2 - 1.$$

Note that

$$J_F(p)v = 2\, v \cdot p.$$

For all $p \in F^{-1}(\{0\})$, the Jacobian $J_F(p)$ is surjective and

$$\mathrm{Ker}(J_F(p)) = \{v \in \mathbb{R}^{n+1} : v \cdot p = 0\}.$$

Moreover,

$$\mathbb{S}^n = F^{-1}(\{0\}).$$

By the previous exercise, we obtain

$$T_p^{\mathrm{ext}}[\mathbb{S}^n] = \mathrm{Ker}(J_F(p)) = \{v \in \mathbb{R}^{n+1} : v \cdot p = 0\},$$

and therefore

$$T^{\mathrm{ext}}[\mathbb{S}^n] = \{(p, v) \in \mathbb{S}^n \times \mathbb{R}^{n+1} : p \cdot v = 0\}.$$

(b) Since n is odd, $n+1$ is even. Define the map

$$\tilde{X} : \mathbb{S}^n \to T^{\mathrm{ext}}[\mathbb{S}^n], \qquad \tilde{X}(p) = \big(p, (p_2, -p_1, p_4, -p_3, \ldots, p_{n+1}, -p_n)\big).$$

It is clear that $\tilde{X}$ is smooth. Moreover,

$$p \cdot (p_2, -p_1, p_4, -p_3, \ldots, p_{n+1}, -p_n) = 0,$$

which implies that

$$(p_2, -p_1, p_4, -p_3, \ldots, p_{n+1}, -p_n) \in T_p^{\mathrm{ext}}[\mathbb{S}^n].$$

Hence, $\tilde{X}(p) \in T^{\mathrm{ext}}[\mathbb{S}^n]$ for all $p \in \mathbb{S}^n$.

Furthermore,

$$(p_2, -p_1, p_4, -p_3, \ldots, p_{n+1}, -p_n) \neq 0 \quad \text{for all } p \in \mathbb{S}^n,$$

so $\tilde{X}$ is a nowhere-vanishing section of $T^{\mathrm{ext}}[\mathbb{S}^n]$.

Finally, let $\Phi : T[\mathbb{S}^n] \to T^{\mathrm{ext}}[\mathbb{S}^n]$ be the diffeomorphism defined in Exercise 1.c, and define

$$X := \Phi^{-1} \circ \tilde{X}.$$

Then X is a smooth, nowhere-vanishing vector field on $\mathbb{S}^n$.

(c) Consider the map

$$\tilde{\Phi} : \mathbb{S}^1 \times \mathbb{R} \to T^{\mathrm{ext}}[\mathbb{S}^1], \qquad \tilde{\Phi}(p, t) = \big(p,\, t(-p_2, p_1)\big).$$

The map $\tilde{\Phi}$ is smooth, and its inverse is given by

$$\tilde{\Phi}^{-1} : T^{\mathrm{ext}}[\mathbb{S}^1] \to \mathbb{S}^1 \times \mathbb{R}, \qquad \tilde{\Phi}^{-1}(p, v) = \big(p,\, (-p_2, p_1) \cdot v\big),$$

which is also smooth. Hence, $\tilde{\Phi}$ is a diffeomorphism.

Moreover, $\tilde{\Phi}$ satisfies

$$\pi\big(\tilde{\Phi}(p,t)\big) = p,$$

and the restriction

$$\tilde{\Phi}\big|_{\{p\}\times\mathbb{R}} : \{p\} \times \mathbb{R} \longrightarrow \{p\} \times T_p^{\mathrm{ext}}[\mathbb{S}^1] = \{p\} \times \{v \in \mathbb{R}^2 : v \cdot p = 0\}$$

$\tilde{\Phi}\big|_{\{p\}\times\mathbb{R}}(t) = (p, t(-p_2, p_1))$ is a linear isomorphism.

Finally, let $\Psi : T[\mathbb{S}^1] \to T^{\mathrm{ext}}[\mathbb{S}^1]$ be the diffeomorphism defined in Exercise 1.c, and define

$$\Phi := \Psi^{-1} \circ \tilde{\Phi}.$$

Then Φ satisfies the desired properties.