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7. Übungsblatt

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Aufgabe 7.1 (2 + 2 + 2 + 2)

Bestimmen Sie für jede der folgenden Abbildungen, ob sie glatt, Homöomorphismen auf ihr Bild, Immersionen oder glatte Einbettungen sind, und begründen Sie Ihre Antwort:

$$\phi : \mathbb{R} \to \mathbb{R}^2, \qquad \phi(t) = (t, |t|).$$
$$\psi : \mathbb{R} \to \mathbb{R}^2, \qquad \psi(t) = (t^2, t^3).$$
$$\eta : \mathbb{R} \to \mathbb{R}^2, \qquad \eta(t) = \left(\frac{t}{2} - \cos t,\ \sin t\right).$$
$$\theta : (-\pi, \pi) \to \mathbb{R}^2, \qquad \theta(t) = (\sin t,\ \sin t \cdot \cos t).$$

Solution.

(a) The map ϕ is a composition of continuous functions, hence continuous. It is injective: if $\phi(t) = \phi(t')$, then $(t, |t|) = (t', |t'|)$, which implies $t = t'$. Its inverse is

$$\phi^{-1} = p_1|_{\phi(\mathbb{R})}, \qquad p_1 : \mathbb{R}^2 \to \mathbb{R}, \quad p_1(x, y) = x,$$

which is continuous. Thus ϕ is a homeomorphism onto its image.

The map ϕ is not smooth. Otherwise, the composition with the smooth function $p_2 : \mathbb{R}^2 \to \mathbb{R}$, $p_2(x, y) = y$, would be smooth. But

$$p_2 \circ \phi(t) = |t|$$

is not differentiable at $t = 0$. Hence ϕ is not smooth.

(b) The map ψ is a composition of smooth functions, so it is smooth. Its inverse is

$$\psi^{-1} = (p_2|_{\psi(\mathbb{R})})^{1/3}, \qquad p_2 : \mathbb{R}^2 \to \mathbb{R}, \quad p_2(x, y) = y,$$

which is continuous as a composition of continuous functions. Therefore, ψ is a homeomorphism onto its image.

The differential of ψ is

$$D\psi_t(v) = \psi'(t)v, \qquad \psi'(t) = (2t, 3t^2).$$

At $t = 0$ we have $D\psi_0 = 0$, which is not injective. Hence ψ is neither an immersion nor a smooth embedding.

(c) The map φ is a composition of smooth functions, hence smooth. Its differential is

$$D\varphi_t(v) = \varphi'(t)v = (1/2 + \sin t,\ \cos t)\, v.$$

Since

$$\|(1/2 + \sin t, \cos t)\|^2 = (1/2 + \sin t)^2 + \cos^2 t \geq \frac{1}{4},$$

we have $(1/2+\sin t, \cos t) \neq 0$ for all t. Thus $D\varphi_t$ is injective and φ is an immersion.
However, φ is not injective. Write

$$\varphi(t) = (\varphi_1(t), \varphi_2(t)), \qquad \varphi_1(t) = \frac{t}{2} - \cos t, \quad \varphi_2(t) = \sin t.$$

Consider the function

$$f : [\pi/2, \pi] \to \mathbb{R}, \qquad f(t) = \varphi_1(3\pi - t) - \varphi_1(t) = \frac{3\pi}{2} - t + 2\cos t,$$

using $\cos(3\pi - t) = -\cos t$. We have

$$f(\pi/2) = \pi, \qquad f(\pi) = \frac{\pi}{2} - 2 < 0.$$

By the intermediate value theorem, there exists $t_0 \in [\pi/2, \pi]$ with $f(t_0) = 0$. Then

$$\varphi_1(3\pi - t_0) = \varphi_1(t_0), \qquad \varphi_2(3\pi - t_0) = \varphi_2(t_0),$$

so $\varphi(t_0) = \varphi(3\pi - t_0)$ with $t_0 < 3\pi - t_0$. Hence φ is an immersion but not injective.

(d) Let $\theta : (-\pi, \pi) \to \mathbb{R}^2$ be given by

$$\theta(t) = (\sin t,\ \sin t \cos t).$$

It is a composition of smooth functions, so it is smooth. Its derivative is

$$\theta'(t) = (\cos t,\ \cos^2 t - \sin^2 t).$$

If $\theta'(t) = 0$, then $\cos t = 0$ and simultaneously $\cos^2 t - \sin^2 t = 0$, which would imply $\sin t = 0$, which is not possible. Thus $\theta'(t) \neq 0$ for all t, and θ is an immersion.
To prove injectivity, suppose $\theta(t) = \theta(t')$. Then $\sin t = \sin t'$ and

$$\sin t \cos t = \sin t' \cos t'.$$

Since $\sin t \neq 0$ for $t \in (-\pi, \pi) \setminus \{0\}$, it follows that $\cos t = \cos t'$. Thus

$$(\cos t, \sin t) = (\cos t', \sin t'),$$

which implies $t = t' + 2\pi n$. Because $t, t' \in (-\pi, \pi)$, we must have $n = 0$, hence $t = t'$. Thus θ is injective.
However, θ^{-1} is not continuous. Consider the sequence

$$a_n = \theta\Big((-1)^n\big(\pi - \tfrac{1}{n}\big)\Big) \in \theta(-\pi, \pi).$$

Then

$$\lim_{n\to\infty} a_n = (0,0) = \theta(0),$$

so (a_n) converges in $\theta(-\pi, \pi)$. But

$$\theta^{-1}(a_n) = (-1)^n\left(\pi - \frac{1}{n}\right)$$

does not converge in $(-\pi, \pi)$. Hence θ is an injective immersion whose inverse is not continuous.

Aufgabe 7.2 (6)

Sei M eine n-dimensionale kompakte differenzierbare Mannigfaltigkeit. Sei $f : M \to \mathbb{R}^d$ stetig, injektiv und eine Immersion. Zeigen Sie, dass f eine Einbettung ist.

Solution. It only remains to show that f is a homeomorphism onto its image. Since f is injective, there exists a map

$$g : f(M) \to M$$

such that $f \circ g = \mathrm{id}_{f(M)}$ and $g \circ f = \mathrm{id}_M$. We now prove that g is continuous.

Because f is smooth and M is compact, f is a closed map. Indeed, if $C \subset M$ is closed, then C is compact, and hence $f(C)$ is compact. Since $\mathbb{R}^n$ is Hausdorff, compact subsets are closed. Therefore $f(C)$ is closed. Thus f sends closed sets to closed sets.

Let $U \subset M$ be an open set. Then $M \setminus U$ is closed, and so $f(M \setminus U)$ is closed in $\mathbb{R}^n$. Hence

$$f(M \setminus U) = \mathbb{R}^n \setminus V$$

for some open set $V \subset \mathbb{R}^n$. We compute

$$g^{-1}(U) = f(M) \setminus g^{-1}(M \setminus U) = f(M) \setminus f(M \setminus U) = f(M) \setminus (\mathbb{R}^n \setminus V) = f(M) \cap V,$$

which is open in $f(M)$. Therefore g is continuous.

Since f is smooth, injective, an immersion, and a homeomorphism onto its image, it follows that f is a smooth embedding.

Aufgabe 7.3 (5)

Sei M eine n-dimensionale differenzierbare Mannigfaltigkeit. Zeigen Sie, dass es eine nicht-konstante Funktion $f \in C^\infty(M, \mathbb{R})$ gibt, sodass $f(x) \neq 0$ für alle $x \in M$ gilt.

Solution. Let us consider the smooth function $g \in C^\infty(\mathbb{R}, \mathbb{R})$ defined by

$$g(t) = \begin{cases} 0, & t \leq 0, \\ \exp(-1/t), & t > 0. \end{cases}$$

For $a < b$ we define the smooth function $g_{a,b} \in C^\infty(\mathbb{R}, \mathbb{R})$ by

$$g_{a,b}(t) = \frac{g(b-t)}{g(t-a) + g(b-t)}.$$

It satisfies

$$g_{a,b}(t) = \begin{cases} 1, & t \leq a, \\ 0, & t \geq b. \end{cases}$$

Let $p, q \in M$ with $p \neq q$. Since M is Hausdorff, there exist open sets U_p, U_q such that

$$U_p \cap U_q = \emptyset, \qquad p \in U_p,\ q \in U_q.$$

Consider the open cover

$$\mathcal{U} = \{U_p, U_q, M \setminus \{p, q\}\}.$$

Choose charts $\phi_p = (V_p, x_p)$ and $\phi_q = (V_q, x_q)$ such that

$$p \in V_p \subset U_p, \qquad q \in V_q \subset U_q,$$

and

$$x_p : V_p \to B_0(r_p) \subset \mathbb{R}^m, \qquad x_q : V_q \to B_0(r_q) \subset \mathbb{R}^m,$$

with

$$x_p(p) = 0, \qquad x_q(q) = 0.$$

Define smooth functions $\varphi_p, \varphi_q : M \to [0,1]$ by

$$\varphi_p(\tilde{p}) = \begin{cases} g_{r_p/16, r_p/4}(\|x_p(\tilde{p})\|^2), & \tilde{p} \in V_p, \\ 0, & \tilde{p} \notin V_p, \end{cases}$$

$$\varphi_q(\tilde{p}) = \begin{cases} g_{r_q/16, r_q/4}(\|x_q(\tilde{p})\|^2), & \tilde{p} \in V_q, \\ 0, & \tilde{p} \notin V_q. \end{cases}$$

These functions are smooth: on V_p and V_q they are compositions of smooth functions, and outside these sets one can find open neighborhoods where they are constant equal 0.

Define $\varphi_1 \in C^\infty(M, [0,1])$ by

$$\varphi_1(\tilde{p}) = 1 - \varphi_p(\tilde{p}) - \varphi_q(\tilde{p}).$$

Then $\{\varphi_p, \varphi_q, \varphi_1\}$ is a smooth partition of unity subordinate to the open cover $\mathcal{U}$. Indeed:

- $$\operatorname{supp}(\varphi_p) \subset U_p, \qquad \operatorname{supp}(\varphi_q) \subset U_q, \qquad \operatorname{supp}(\varphi_1) \subset M \setminus \{p, q\}.$$
- For every $\tilde{p} \in M$,
$$\varphi_p(\tilde{p}) + \varphi_q(\tilde{p}) + \varphi_1(\tilde{p}) = 1.$$

Define $f \in C^\infty(M, \mathbb{R})$ by

$$f = \frac{1}{2}\varphi_p + \varphi_q + \varphi_1.$$

For every $\tilde{p} \in M$ we have

$$f(\tilde{p}) = 1 - \frac{1}{2}\varphi_p(\tilde{p}) \geq \frac{1}{2}.$$

Moreover,

$$f(p) = \frac{1}{2}, \qquad f(q) = 1,$$

so f is not constant.

Note.One can (essentially in the same way as in the solution of the previous exercise, though with more technical details) prove the following important fact:

Fact. Let M be a smooth manifold. For every open cover $\mathcal{U} = \{U_\alpha\}$ of M, there exists a smooth partition of unity $\{\varphi_\alpha\} \subset C^\infty(M, [0,1])$ subordinate to $\mathcal{U}$.

Aufgabe 7.4 (3 + 2)

Seien N und M differenzierbare Mannigfaltigkeiten und $f : N \to M$ eine Einbettung.

(a) Zeigen Sie, dass $D_q[f] : T_q[N] \to T_{f(p)}[M]$ injektiv ist.

(b) Zeigen Sie, dass die Abbildung

$$D[f] : T[N] \to T[M], \qquad D[f](q,v) = (f(q), D_q[f](v)),$$

eine Einbettung ist.

Solution.

(a) Consider charts $\phi = (U, x)$ of N around $q \in N$ and $\psi = (V, y)$ of M around $f(q) \in M$. We have the isomorphisms

$$\Theta_{\phi,q} : T_q[N] \to \mathbb{R}^n, \quad \Theta_{\phi,q}[\gamma] = \frac{d}{dt}(x \circ \gamma)\Big|_{t=0},$$

$$\Theta_{\psi,f(q)} : T_{f(q)}[M] \to \mathbb{R}^m, \quad \Theta_{\psi,f(q)}[\gamma] = \frac{d}{dt}(y \circ \gamma)\Big|_{t=0}.$$

Then the differential of f at q can be expressed as

$$D_q[f] = \Theta^{-1}_{\psi,f(q)} \circ J_{y\circ f\circ x^{-1}}(x(q)) \circ \Theta_{\phi,q}.$$

Since f is an embedding, it is in particular an immersion, so $J_{y\circ f\circ x^{-1}}(x(q))$ is injective. Being a composition of injective maps, $D_q[f]$ is also injective. Moreover, note that for $[\gamma] \in D_q[f](T_q[N]) = \Theta^{-1}_{\psi,f(q)}(\mathrm{Ran}(J_{y\circ f\circ x^{-1}}(x(q)))$ one has

$$(D_q[f])^{-1}[\gamma] = \Theta^{-1}_{\psi,q} \circ (J_{y\circ f\circ x^{-1}}(x(q))^{\dagger} \circ \Theta_{\psi,f(q)}[\gamma] \tag{1}$$

where for matrix $M \in \mathcal{M}_{m\times n}(\mathrm{R})$ with rank n the matrix $M^{\dagger}$ denotes its pseudoinverse

$$M^{\dagger} = (M^T M)^{-1} M^T.$$

That satisfies, $M^{\dagger} M v = v$ for all $v \in \mathbb{R}^n$.

(b) Note that $D[f]$ is injective. Indeed, if $D[f](q, v) = D[f](q', v')$, then $f(q) = f(q')$ and $D_q[f]v = D_{q'}[f]v'$. Since f and $D_q[f]$ are injective, it follows that $q = q'$ and $v = v'$.

Moreover, the inverse of $D[f]$ is given by

$$(D[f])^{-1} : D[f](T[N]) \to T[N], \qquad (D[f])^{-1}(p, [\gamma]) = \Big(f^{-1}(p), (D_{f^{-1}(p)}[f])^{-1}[\gamma]\Big),$$

where

$$D[f](T[N]) = \bigcup_{p\in f(N)} \{p\} \times D_{f^{-1}(p)}[f](T_{f^{-1}(p)}[N]).$$

To prove continuity of $D[f]^{-1}$, take $(p_0, [\gamma_0]) \in D[f](T[N])$. There exist charts $\phi = (x, U)$ of N around $f^{-1}(p_0)$ and $\psi = (y, V)$ of M around p_0 such that

$$V \cap f(N) \subset f(U).$$

Consider the homeomorphisms

$$\Theta_\phi : T[U] \to x(U) \times \mathbb{R}^n, \quad \Theta_\phi(q,[\gamma]) = (x(q), \Theta_{\phi,q}[\gamma]), \tag{2}$$

$$\Theta_\psi : T[V] \to y(V) \times \mathbb{R}^m, \quad \Theta_\psi(p,[\gamma]) = (y(p), \Theta_{\psi,p}[\gamma]). \tag{3}$$

Then

$$\Theta_\phi \circ D[f]^{-1}| \circ \Theta_\psi^{-1}| : \Theta_\psi(T[V] \cap D[f](T[N])) \to x(U) \times \mathbb{R}^n$$

is explicitly

$$\begin{aligned}\Theta_\phi \circ D[f]^{-1}| \circ \Theta_\psi^{-1}|(a,v) &= \Theta_\phi \circ D[f]^{-1}|\big(y^{-1}(a), \Theta^{-1}_{\psi,y^{-1}(a)}(v)\big)\\ &= \Theta_\psi(f^{-1} \circ y^{-1}(a), (D_{f^{-1}\circ y^{-1}(a)}[f])^{-1}\Theta^{-1}_{\psi,y^{-1}(a)}(v))\\ &= (x \circ f^{-1} \circ y^{-1}(a), \Theta_{\psi,f^{-1}\circ y^{-1}(a)}(D_{f^{-1}\circ y^{-1}(a)}[f])^{-1}\Theta^{-1}_{\psi,y^{-1}(a)}(v))\\ &= (x \circ f^{-1} \circ y^{-1}(a), \Big(J_{y\circ f\circ x^{-1}}(x \circ f^{-1} \circ y^{-1}(a))\Big)^{\dagger} v)\end{aligned}$$

where in the last equality we use Eq. (1). The last expression implies that the composition is continuous. Therefore, (since Θ_ψ, Θ_ϕ are homeomorphisms) $D[f]^{-1}|_{T[V]\cap D[f](T[N])}$ is continuous. Since $T[V] \cap D[f](T[N])$ is an open of $D[f](T[N])$ with $(p_0,[\gamma_0]) \in T[V] \cap D[f](T[N])$ this proves that $(D[f])^{-1}$ is continuous.

For smoothness, consider charts $\psi = (V,y)$ of M and $\phi = (U,x)$ of N and the local maps as in Eqs. (2), (3). Then the map

$$\Theta_\psi \circ D[f]| \circ \Theta_\phi^{-1} : \Theta_\phi(T[U] \cap (D[f])^{-1}(T[V])) \subset U \times \mathbb{R}^n \to V \times \mathbb{R}^m$$

is given by

$$\Theta_\psi \circ D[f]| \circ \Theta_\phi^{-1}(a,v) = \big(y \circ f \circ x^{-1}(a), \Theta_{\psi,f\circ x^{-1}(a)}\, D_{x^{-1}(a)}[f]\, (\Theta_{\phi,x^{-1}(a)})^{-1}(v)\big).$$

Using

$$D_{x^{-1}(a)}[f] = \Theta^{-1}_{\psi,f(x^{-1}(a))} \circ J_{y\circ f\circ x^{-1}}(a) \circ \Theta_{\phi,x^{-1}(a)},$$

we obtain

$$\Theta_\psi \circ D[f]| \circ \Theta_\phi^{-1}(a,v) = \big(y \circ f \circ x^{-1}(a), J_{y\circ f\circ x^{-1}}(a)\, v\big),$$

which is smooth. Moreover, its Jacobian is

$$J_{\Theta_\psi\circ D[f]|\circ\Theta_\phi^{-1}}(a,v) = \begin{pmatrix} J_{y\circ f\circ x^{-1}}(a) & 0 \\ M(a,v) & J_{y\circ f\circ x^{-1}}(a)\end{pmatrix}.$$

Since $J_{y\circ f\circ x^{-1}}(a)$ is injective, the total Jacobian is injective. Therefore, $D[f]$ is an immersion, and being injective with continuous inverse, it is an embedding.