Vektoranalysis
Wintersemester 25/26
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6. Übungsblatt

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Aufgabe 6.1 (8)

Sei $\Omega \subset \mathbb{R}$ eine offene Menge. Für $n \geq 2$ sei $f \in C^2(\Omega, \mathbb{R}^n)$. Zeigen Sie, dass für jedes $\varepsilon > 0$ ein $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ mit $\max\{|v_1|, \cdots, |v_n|\} < \varepsilon$ existiert, sodass die Funktion

$$f_v(t) := f(t) + vt$$

für alle $t \in \Omega$ die Eigenschaft $f_v'(t) \neq 0$ erfüllt.

Solution Let us denote $f(t) = (f_1(t), \dots, f_n(t))$. Fix a closed interval $[a, b] \subset \Omega$. Since $f' \in C^1(\Omega, \mathbb{R}^n)$, there exists $L > 0$ such that for all $x, y \in [a, b]$ one has

$$|f_j'(x) - f_j'(y)| \leq L|x - y|, \qquad j \in \{1, \dots, n\}.$$

Let $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_p = b\}$ be a partition of $[a, b]$. For each interval $[t_{i-1}, t_i]$ define

$$m_{i,j} := \min\{f_j'(x) : x \in [t_{i-1}, t_i]\}, \qquad M_{i,j} := \max\{f_j'(x) : x \in [t_{i-1}, t_i]\},$$

and consider the box

$$Q_i = [m_{i,1}, M_{i,1}] \times \cdots \times [m_{i,n}, M_{i,n}].$$

Then $f'([t_{i-1}, t_i]) \subset Q_i$. Moreover,

$$|M_{i,j} - m_{i,j}| \leq L|t_i - t_{i-1}|.$$

Hence,

$$\lambda^n(Q_i) \leq L^n |t_i - t_{i-1}|^n = L^n |t_i - t_{i-1}| \, |t_i - t_{i-1}|^{n-1}.$$

Therefore,

$$\lambda^n(f'([a, b])) \leq \sum_{i=1}^{p} \lambda^n(Q_i) \leq L^n \max_{i \in \{1, \dots, p\}} |t_i - t_{i-1}|^{n-1} \sum_{i=1}^{p} (t_i - t_{i-1}) = L^n (b - a) \max_i |t_i - t_{i-1}|^{n-1}.$$

Since $n - 1 \geq 1$, for every $\varepsilon > 0$ we may choose a partition $\mathcal{P}$ such that the right-hand side is smaller than ε. Thus,

$$\lambda^n(f'([a, b])) = 0.$$

Now take a countable family of closed intervals $\{I_k\}$ such that $\Omega = \bigcup_k I_k$. Then

$$\lambda^n(f'(\Omega)) = 0.$$

Hence,

$$B(0, \varepsilon) \cap \left(\mathbb{R}^n \setminus (-f'(\Omega))\right) \neq \emptyset.$$

Choose v in this set. Then $|v_j| < \varepsilon$ and $v \notin -f'(\Omega)$, which means $v \neq -f'(x)$ for all $x \in \Omega$. Therefore

$$f_v'(x) = f'(x) + v \neq 0 \qquad \text{for all } x \in \Omega.$$

This proves the claim.

Aufgabe 6.2 (2 + 2 + 2 + 2)

1. Gib eine differenzierbare Abbildung
$$f : \mathbb{R}^2 \to \mathbb{R}^3$$
an, sodass $\mathbb{R}^2/2\pi\mathbb{Z}^2$ und $\operatorname{ran}[f]$ diffeomorph sind.

2. Zeige, dass eine n-dimensionale Mannigfaltigkeit M, die ein Produkt von Sphären ist, eine Einbettung in $\mathbb{R}^{n+1}$ besitzt:
   (a) Zeige, dass die Abbildung
   $$(t, x) \mapsto e^t x$$
   eine Einbettung von $\mathbb{R} \times \mathbb{S}^d$ nach $\mathbb{R}^{d+1}$ ist.
   (b) Zeige, dass $\mathbb{S}^d \times \mathbb{R}^k$ in $\mathbb{R}^{d+k}$ eingebettet werden kann.
   (c) Sei $d_1, \dots, d_k \geq 1$ mit $d_1 + \cdots + d_k = n$. Zeige, dass eine Einbettung existiert:
   $$f : \mathbb{S}^{d_1} \times \cdots \times \mathbb{S}^{d_k} \to \mathbb{R}^{n+1}.$$

Solution.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the function
$$f(\theta_1, \theta_2) = \big((2 + \cos\theta_2)\cos\theta_1,\ (2 + \cos\theta_2)\sin\theta_1,\ \sin\theta_2\big).$$
Note that if $(\theta_1, \theta_2) = (\tilde\theta_1, \tilde\theta_2) + 2\pi(n_1, n_2)$, then $f(\theta_1, \theta_2) = f(\tilde\theta_1, \tilde\theta_2)$. Therefore, there exists a continuous map
$$\tilde f : \mathbb{T}^2 \to \mathbb{R}^3, \qquad \tilde f[\theta_1, \theta_2] = f(\theta_1, \theta_2).$$
Moreover, if $\tilde f[\theta_1, \theta_2] = \tilde f[\varphi_1, \varphi_2]$, then $(\cos\theta_1, \sin\theta_1) = (\cos\varphi_1, \sin\varphi_1)$, so $\theta_1 = \varphi_1 + 2\pi n_1$. Similarly $\theta_2 = \varphi_2 + 2\pi n_2$. Thus $[\theta_1, \theta_2] = [\varphi_1, \varphi_2]$ and $\tilde f$ is injective. Since $\mathbb{T}^2$ is compact and $\mathbb{R}^3$ is Hausdorff, any injective continuous map is a homeomorphism onto its image.
For $[\theta_1, \theta_2] \in \mathbb{T}^2$ consider the local parametrization
$$\psi : (\theta_1 - \pi,\ \theta_1 + \pi) \times (\theta_2 - \pi,\ \theta_2 + \pi) \longrightarrow \pi\big((\theta_1 - \pi, \theta_1 + \pi) \times (\theta_2 - \pi, \theta_2 + \pi)\big),$$
$$\psi(\theta, \sigma) = [\theta, \sigma].$$
Then
$$\tilde f \circ \psi(\theta, \sigma) = f(\theta, \sigma),$$
and
$$J_{\tilde f \circ \psi}(\theta, \sigma) = J_f(\theta, \sigma) = \begin{pmatrix} -(2 + \cos\sigma)\sin\theta & -\sin\sigma\cos\theta \\ (2 + \cos\sigma)\cos\theta & -\sin\sigma\sin\theta \\ 0 & \cos\sigma \end{pmatrix}.$$
The column vectors C_1, C_2 satisfy $C_1 \cdot C_2 = 0$ and $\|C_1\|, \|C_2\| > 0$, hence they are linearly independent. Thus $J_f(\theta, \sigma)$ is an immersion. Therefore $\tilde f$ is a smooth embedding, and hence
$$\mathbb{T}^2 \cong \tilde f(\mathbb{T}^2) = f(\mathbb{R}^2).$$

2. Consider the map
$$f : \mathbb{R} \times \mathbb{S}^d \to \mathbb{R}^{d+1}, \qquad f(t,x) = e^t x.$$
This map is clearly continuous. If $f(t,x) = f(s,y)$, then $\|f(t,x)\| = e^t = e^s$, hence $t = s$, and then $x = y$. Thus f is injective. Its inverse is
$$g : \mathbb{R}^{d+1} \setminus \{0\} \to \mathbb{R} \times \mathbb{S}^d, \qquad g(z) = (\log \|z\|,\ z/\|z\|),$$
which is continuous. Thus f is a homeomorphism onto its image.
Let $(x_1, \dots, x_{d+1}) \in \mathbb{S}^d$. Assume $x_1 > 0$ and consider the local chart
$$\varphi : U \subset \mathbb{S}^d \to \mathbb{R}^d, \qquad \varphi(x_1, \dots, x_{d+1}) = (x_2, \dots, x_{d+1}),$$
with inverse
$$\varphi^{-1}(x_2, \dots, x_{d+1}) = \big(\sqrt{1 - \|x\|^2},\ x_2, \dots, x_{d+1}\big).$$
Then
$$f \circ (\mathrm{id}_{\mathbb{R}} \times \varphi^{-1})(t, x_2, \dots, x_{d+1}) = e^t \big(\sqrt{1 - \|x\|^2},\ x_2, \dots, x_{d+1}\big),$$
which is smooth and whose Jacobian matrix is
$$J_{f \circ (\mathrm{id} \times \varphi^{-1})} = e^t \begin{pmatrix} \sqrt{1 - \|x\|^2} & -\frac{x_2}{\sqrt{1-\|x\|^2}} & \cdots & -\frac{x_{d+1}}{\sqrt{1-\|x\|^2}} \\ x_2 & 1 & 0 & \cdots \\ \vdots & & \ddots & \vdots \\ x_{d+1} & 0 & \cdots & 1 \end{pmatrix},$$
If $J_{f \circ (\mathrm{id} \times \varphi^{-1})} v = 0$, then $v_i = -x_i v_1$ and
$$0 = (v_1, v_2, \cdots, v_{d+1}) \cdot \big(\sqrt{1 + \|x\|^2}, \cdots, -(1 + \|x\|^2)^{-1/2} x_{d+1}\big) = v_1 \frac{1}{\sqrt{1 - \|x\|^2}},$$
then $v_1 = 0$ and $v_2 = v_3 = \cdots = v_{d+1} = 0$. Hence $J_{f \circ (\mathrm{id} \times \varphi^{-1})}$ is injective. Thus f is a smooth embedding.
Now define
$$h : \mathbb{S}^d \times \mathbb{R}^k \to (\mathbb{S}^d \times \mathbb{R}) \times \mathbb{R}^{k-1}, \qquad h(x, (r_1, \dots, r_k)) = ((x, r_1), (r_2, \dots, r_k)),$$
and
$$F :\ (\mathbb{S}^d \times \mathbb{R}) \times \mathbb{R}^{k-1} \to \mathbb{R}^{d+1} \times \mathbb{R}^{k-1}, \qquad F((x,r), (r_2, \dots, r_k)) = (e^r x,\ (r_2, \dots, r_k)).$$
Since the map in part (a) is an embedding, F is an embedding. It is clear that h is also an embedding; hence $F \circ h$ is an embedding. Thus $\mathbb{S}^d \times \mathbb{R}^k$ embeds in $\mathbb{R}^{d+k}$.
Finally, for products of spheres we proceed by induction. The base case $\mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1}$ is clear. Assume
$$F : \mathbb{S}^{d_1} \times \cdots \times \mathbb{S}^{d_k} \to \mathbb{R}^{n+1}, \qquad n = d_1 + \cdots + d_k,$$
is an embedding. Let $d_{k+1} \geq 1$. By (b) there is an embedding
$$G : \mathbb{R}^{n+1} \times \mathbb{S}^{d_{k+1}} \to \mathbb{R}^{n+d_{k+1}+1}.$$
Define
$$H : \big(\mathbb{S}^{d_1} \times \cdots \times \mathbb{S}^{d_k}\big) \times \mathbb{S}^{d_{k+1}} \to \mathbb{R}^{n+d_{k+1}+1}, \qquad H(x,y) = G\big(F(x),\ y\big).$$
Since H is a composition of embeddings, it is an embedding. This completes the induction.

Aufgabe 6.3 (2 + 2 + 2 + 2)

Wir bezeichnen mit $\mathbb{S}^2$ die Einheitskugel in $\mathbb{R}^3$ und definieren die folgende Äquivalenzrelation auf $\mathbb{S}^2$:

$$x \sim y \iff x = \pm y.$$

Sei $P^2 = \mathbb{S}^2 / \sim$. Betrachten wir die Abbildung

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \quad (x, y, z) \mapsto (2xz,\, 2yz,\, 1 - 2z^2).$$

(a) Zeigen Sie, dass die Einschränkung von $g = f|_{\mathbb{S}^2}$ auf $\mathbb{S}^2$ eine C^∞-Abbildung von $\mathbb{S}^2$ nach $\mathbb{S}^2$ ist.

(b) Bestimmen Sie die Punkte $q \in \mathbb{S}^2$, an denen die Ableitung $D_q[g]$ nicht invertierbar ist.

(c) Zeigen Sie, dass eine glatte Abbildung
$$h : P^2 \to \mathbb{S}^2$$
existiert, sodass
$$g = h \circ \pi,$$
wobei $\pi : \mathbb{S}^2 \to P^2$ die kanonische Projektion ist.

(d) Zeigen Sie, dass h surjektiv ist, und bestimmen Sie die Punkte $q \in P^2$, an denen die Ableitung $D_q[h]$ nicht invertierbar ist.

Solution

(a) If $(x, y, z) \in \mathbb{S}^2$, then

$$\|f(x, y, z)\|^2 = 4x^2z^2 + 4y^2z^2 + 1 - 4z^2 + 4z^4 = 4z^2(x^2 + y^2 + z^2) - 4z^2 + 1 = 1.$$

Thus $f(\mathbb{S}^2) \subset \mathbb{S}^2$. Since f is a polynomial map, it is C^∞.

For surjectivity, note first that $f(1, 0, 0) = (0, 0, 1)$. If $z \neq 1$, then

$$f\left(\frac{\sqrt{2}}{2\sqrt{1-z}}x,\ \frac{\sqrt{2}}{2\sqrt{1-z}}y,\ \sqrt{\frac{1-z}{2}}\right) = (x, y, z).$$

(b) Let $(x_0, y_0, z_0) \in \mathbb{S}^2$.

(a) If $z_0 = 0$, assume w.l.o.g. $x_0 > 0$. Take the local parametrizations

$$\phi(y, z) = (\sqrt{1 - y^2 - z^2}, y, z), \qquad \psi(x, y) = (x, y, \sqrt{1 - x^2 - y^2}), \tag{1}$$

with $\psi^{-1}(x, y, z) = (x, y)$. Then

$$\psi^{-1} \circ g \circ \phi(y, z) = (2z\sqrt{1 - y^2 - z^2},\, 2yz).$$

The Jacobian at $(y_0, 0)$ is $\begin{pmatrix} 0 & 2\sqrt{1 - y_0^2} \\ 0 & 2y_0 \end{pmatrix}$ and is not invertible.

(b) Suppose $z_0 \neq 0$. If $(x_0, y_0, z_0) \neq (0,0,\pm 1)$ and w.l.o.g. $x_0 \cdot z_0 > 0$, use the local parametrization
$$\phi(y,z) = (\sqrt{1-y^2-z^2}, y, z).$$
with $\phi^{-1}(x,y,z) = (y,z)$. Then, $\phi^{-1} \circ g \circ \phi(y,z) = (2yz,\ 1-2z^2)$. The Jacobian is at (y_0, z_0) is $\begin{pmatrix} 2z_0 & 2y_0 \\ 0 & -4z_0 \end{pmatrix}$. Since $z_0 \neq 0$, the Jacobian is invertible.

For $(0,0,1)$ we have $g(0,0,1) = (0,0,-1)$. Use the local parametrizations
$$\phi(x,y) = (x, y, \sqrt{1-x^2-y^2}), \qquad \psi(x,y) = (x, y, -\sqrt{1-x^2-y^2}), \tag{2}$$
with $\psi^{-1}(x,y,z) = (x,y)$, giving
$$\psi^{-1} \circ g \circ \phi(x,y) = (2x\sqrt{1-x^2-y^2},\ 2y\sqrt{1-x^2-y^2}),$$
whose Jacobian at $(0,0)$ is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and it is invertible. A similar computation works at $(0,0,-1)$.

Thus the critical points are exactly the set
$$\{(x,y,0) \in \mathbb{S}^2\}.$$

(c) If $(x,y,z) \sim (x',y',z')$ in P^2 then $(x,y,z) = \pm(x',y',z')$, hence $g(x,y,z) = g(x',y',z')$. Thus there exists a continuous
$$h : P^2 \to \mathbb{S}^2$$
such that $g = h \circ \pi$.

Local charts for P^2 may be taken as:
$$\phi_x : \{(x,y,z) : x \neq 0\}/\sim\ \to B_1(0), \qquad \phi_x[(x,y,z)] = (y,z),$$
$$\phi_z : \{(x,y,z) : z \neq 0\}/\sim\ \to B_1(0), \qquad \phi_z[(x,y,z)] = (x,y),$$
with inverses
$$\phi_x^{-1}(y,z) = [(\sqrt{1-y^2-z^2}, y, z)], \qquad \phi_z^{-1}(x,y) = [(x, y, \sqrt{1-x^2-y^2})].$$
Let $[(x_0,y_0,z_0)] \in \mathbb{T}^2$. If $z_0 = 0$ and $x_0 \neq 0$, then $[(x_0,y_0,0)]$ satisfies $h[(x_0,y_0,0)] = (0,0,1)$. We consider $\psi^{-1} \circ h \circ \phi_x^{-1}$ with ψ as in Eq. (1)
$$\psi^{-1} \circ h \circ \phi_x^{-1}(y,z) = \psi^{-1} \circ g\left(\sqrt{1-y^2-z^2}, y,\ z\right) = (2z\sqrt{1-y^2-y^2}, 2yz),$$
The Jacobian at $(y_0, 0)$ is not invertible. The case $y_0 \neq 0$ is similar.

If $z_0 \neq 0$. If $[(x_0,y_0,z_0)] \neq [(0,0,1)]$. Assuming w.l.o.g. that $x_0 \cdot z_0 > 0$, we obtain (taking ϕ as in Eq. (1))
$$\phi^{-1} \circ h \circ \phi_x^{-1}(y,z) = \phi^{-1} \circ g(\sqrt{1-y^2-z^2}, y, z) = (2yz, 1-2z^2),$$
and its Jacobian at (y_0, z_0) is invertible. Finally for $[(0,0,1)]$, taking ψ as in Eq. (2) we obtain
$$\psi^{-1} \circ h \circ \phi_z^{-1}(x,y) = (2x\sqrt{1-x^2-y^2}, 2y\sqrt{1-x^2-y^2}).$$
The Jacobian at $(0,0)$ is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ which is invertible. The critical points is the set $\{[(x,y,0)] \in P^2\}$.