Technische Universität Braunschweig

5. Übungsblatt

Upload: 17.11.2025
Deadline: 25.11.2025, 11.30 Uhr (vor der Übung).

Aufgabe 5.1 (4 + 4)

(a) Bestimmen Sie einen Atlas $\{\varphi_+, \varphi_-\}$ der Sphäre S^2 mittels der stereographischen Projektionen auf die reelle Ebene $\mathbb{R}^2$.

(b) Zeigen Sie, dass die so erhaltene Mannigfaltigkeit dieselbe ist wie diejenige, die man durch die Gleichung

$$\|\vec{x}\| = 1, \qquad \vec{x} \in \mathbb{R}^3$$

als eingebettete Untermannigfaltigkeit erhält.

Solution

(a) For the open sets $U_\pm := \mathbb{S}^2 \setminus \{(0,0,\pm 1)\}$, consider the stereographic projection

$$\phi_\pm : U_\pm \to \mathbb{R}^2, \qquad \phi_\pm(x_1,x_2,x_3) = \frac{1}{1 \mp x_3}(x_1,x_2).$$

This is a continuous function, with inverse

$$\phi_\pm^{-1} : \mathbb{R}^2 \to U_\pm, \qquad \phi_\pm^{-1}(x_1,x_2) = \left(\frac{2x_1}{x_1^2+x_2^2+1}, \frac{2x_2}{x_1^2+x_2^2+1}, \pm 1 \mp \frac{2}{x_1^2+x_2^2+1} \right).$$

The change of charts is given by

$$\phi_+ \circ \phi_-^{-1} = \phi_- \circ \phi_+^{-1} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}, \qquad (\phi_- \circ \phi_+^{-1})(x_1,x_2) = \frac{1}{x_1^2+x_2^2}(x_1,x_2),$$

which is smooth. Hence, $\{\phi_+, \phi_-\}$ defines an atlas on $\mathbb{S}^2$.

(b) Consider the map $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x) = \|x\|^2$. Let $y \in \mathbb{R}^3$ with $f(y) = 1$. Note that $\nabla f(y) = 2y \neq 0$. Then, one of the entries of y is nonzero.

Case $y_1 \neq 0$. By the implicit function theorem, there exist an open set $U \subset \mathbb{R}^2$ with $(y_2,y_3) \in U$ and a function

$$\varphi : U \subset \mathbb{R}^2 \to \mathbb{R},$$

such that $\varphi(y_2,y_3) = y_1$ and

$$f(\varphi(x_2,x_3),x_2,x_3) = 1.$$

Define

$$\phi : U \subset \mathbb{R}^2 \to \mathbb{S}^2, \qquad \phi(x_2,x_3) = (\varphi(x_2,x_3),x_2,x_3),$$

which is a homeomorphism onto its image with inverse

$$\phi^{-1} : \phi(U) \subset \mathbb{S}^2 \to U \subset \mathbb{R}^2, \qquad \phi^{-1}(x_1,x_2,x_3) = (x_2,x_3).$$

Thus, $(\phi(U), \phi^{-1})$ gives a chart of the manifold.

To prove compatibility with $\phi_\pm$, consider

$$\phi_+ \circ \phi : U \cap \phi^{-1}(\mathbb{S}^2 \setminus \{(0,0,1)\}) \to \phi_+(\phi(U) \cap (\mathbb{S}^2 \setminus \{(0,0,1)\})),$$

$$(\phi_+ \circ \phi)(x_2, x_3) = \phi_+(\varphi(x_2,x_3), x_2, x_3) = \frac{1}{1-x_3}(\varphi(x_2,x_3), x_2).$$

Note that

$$\mathbb{R} \times \{1\} \cap \phi^{-1}(\mathbb{S}^2 \setminus \{(0,0,1)\}) = \emptyset,$$

for otherwise $\phi(x_2, 1) \in \mathbb{S}^2 \setminus \{(0,0,1)\}$ would imply

$$\varphi(x_2,1)^2 + x_2^2 + 1 = 1 \quad \Rightarrow \quad x_2^2 = 0 = \varphi(x_2,1)^2,$$

hence $\phi(x_2, 1) = (0,0,1) \in \mathbb{S}^2 \setminus \{(0,0,1)\}$, a contradiction. Thus, $\phi_+ \circ \phi$ is smooth.

For the other composition,

$$\phi^{-1} \circ \phi_+^{-1}(x_2, x_3) = \left(\frac{2x_2}{x_2^2 + x_3^2 + 1}, 1 - \frac{2}{x_2^2 + x_3^2 + 1} \right),$$

which is also smooth. The compatibility with ϕ_- is analogous.

Case $y_3 \neq 0$. In this case, the chart for (y_1, y_2, y_3) is given by $(\phi(U), \phi^{-1})$, where

$$\phi^{-1} : \phi(U) \subset \mathbb{S}^2 \to U \subset \mathbb{R}^2, \qquad \phi^{-1}(x_1, x_2, x_3) = (x_1, x_2),$$

and

$$\phi : U \to \mathbb{S}^2, \qquad \phi(x_1, x_2) = (x_1, x_2, \varphi(x_1, x_2)).$$

The change of charts is given by

$$\phi_+ \circ \phi : U \cap \phi^{-1}(\mathbb{S}^2 \setminus \{(0,0,1)\}) \to \phi_+(\phi(U) \cap (\mathbb{S}^2 \setminus \{(0,0,1)\})),$$

$$(\phi_+ \circ \phi)(x_1, x_2) = \frac{1}{1 - \varphi(x_1, x_2)}(x_1, x_2).$$

If $\varphi(x_1, x_2) = 1$ for some $(x_1, x_2) \in \phi^{-1}(\mathbb{S}^2 \setminus \{(0,0,1)\})$, then $(x_1, x_2) = (0,0)$ and $\phi(0,0) = (0,0,1) \in \mathbb{S}^2 \setminus \{(0,0,1)\}$, which is impossible. Thus, $\phi_+ \circ \phi$ is smooth.

Finally,

$$\phi^{-1} \circ \phi_+^{-1}(x_1, x_2) = \frac{2}{x_1^2 + x_2^2 + 1}(x_1, x_2),$$

which is smooth.

Aufgabe 5.2 (8)

Zeigen Sie, dass $\mathbb{R}^d$ eine Überlagerungsmannigfaltigkeit (Covering Manifold) des d-dimensionalen Torus

$$\mathbb{T}^d = \mathbb{R}^d / 2\pi\mathbb{Z}^d$$

ist.

Solution

Let

$$\pi : \mathbb{R}^d \to \mathbb{R}^d/2\pi\mathbb{Z}^d =: \mathbb{T}^d, \qquad \pi(x) = [x]$$

be the quotient map. Fix $p \in \mathbb{T}^d$ and choose any representative $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ with $[x] = p$. Consider the open rectangle

$$U_x = (x_1 - \pi, x_1 + \pi) \times \cdots \times (x_d - \pi, x_d + \pi) \subset \mathbb{R}^d.$$

We claim that

$$\pi^{-1}\big(\pi(U_x)\big) = \bigcup_{n\in\mathbb{Z}^d} (U_x + 2\pi n), \qquad U_x + 2\pi n := \{y + 2\pi n : \ y \in U_x\}. \tag{1}$$

"$\subset$". If $y \in \pi^{-1}(\pi(U_x))$, then $\pi(y) \in \pi(U_x)$. Hence there exists $z \in U_x$ with $[y] = \pi(y) = \pi(z) = [z]$, so $y = z + 2\pi n$ for some $n \in \mathbb{Z}^d$. Thus $y \in U_x + 2\pi n$.

"$\supset$". If $y \in U_x + 2\pi n$ for some $n \in \mathbb{Z}^d$, then $y = z + 2\pi n$ for some $z \in U_x$. Hence $\pi(y) = [y] = [z] \in \pi(U_x)$, so $y \in \pi^{-1}(\pi(U_x))$.

Therefore (??) holds, and since the right-hand side is a union of open sets, $\pi(U_x) \subset \mathbb{T}^d$ is open.

Next we show the translates are pairwise disjoint: if $n \neq m$ in $\mathbb{Z}^d$ then

$$(U_x + 2\pi n) \cap (U_x + 2\pi m) = \varnothing.$$

Indeed, suppose y lies in the intersection. Then for each coordinate j,

$$y_j - 2\pi n_j,\ y_j - 2\pi m_j \in (x_j - \pi, x_j + \pi),$$

so

$$|2\pi(n_j - m_j)| = |(y_j - 2\pi n_j) - (y_j - 2\pi m_j)| < (x_j + \pi) - (x_j - \pi) = 2\pi,$$

which forces $n_j = m_j$. Hence $n = m$, a contradiction.

Finally, we show $\pi(U_x)$ is diffeomorphic to U_x. Define

$$\Phi : \bigcup_{n\in\mathbb{Z}^d} (U_x + 2\pi n) \to U_x, \qquad \Phi(y) = y - 2\pi n \quad \text{for } y \in U_x + 2\pi n.$$

The map Φ is continuous (indeed smooth) and if $[y'] = [y]$ then $\Phi(x) = \Phi(y)$, hence there exists a continuous map

$$\widetilde{\Phi} : \pi(U_x) \to U_x$$

satisfying $\widetilde{\Phi} \circ \pi = \Phi$. On the other hand, $\pi|_{U_x} : U_x \to \pi(U_x)$ is surjective and $\widetilde{\Phi}$ is its inverse:

$$\widetilde{\Phi} \circ \pi|_{U_x} = \mathrm{id}_{U_x}, \qquad \pi|_{U_x} \circ \widetilde{\Phi} = \mathrm{id}_{\pi(U_x)}.$$

Thus $\pi|_{U_x} : U_x \to \pi(U_x)$ is a homeomorphism, since all maps involved are smooth, it is a diffeomorphism onto the open subset $\pi(U_x) \subset \mathbb{T}^d$.

Aufgabe 5.3 (2 + 2 + 2 + 2)

Seien $0 < r < R$ und betrachten wir die Abbildungen $f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$, $F : \mathbb{R}^3 \to \mathbb{R}$,

$$f((x_1, y_1), (x_2, y_2)) = \big((R + rx_2)x_1,\ (R + rx_2)y_1,\ ry_2\big),$$

$$F(x, y, z) = 4(x^2 + y^2)R^2 - \big(x^2 + y^2 + z^2 + R^2 - r^2\big)^2.$$

(a) Zeigen Sie, dass f glatt und injektiv ist.

(b) Zeigen Sie, dass f eine glatte Einbettung ist.

(c) Zeigen Sie, dass für alle $x \in F^{-1}(0)$ gilt: $\nabla F(x) \neq 0$.

(d) Zeigen Sie, dass
$$F^{-1}(0) = f(\mathbb{S}^1 \times \mathbb{S}^1).$$

Solution

(a) Let us show that f is injective. If
$$a = ((x_1, y_1), (x_2, y_2)) = ((\tilde{x}_1, \tilde{y}_1), (\tilde{x}_2, \tilde{y}_2)) = \tilde{a}$$
and $f(a) = f(\tilde{a})$, then the third coordinate gives
$$ry_2 = r\tilde{y}_2 \quad \Rightarrow \quad y_2 = \tilde{y}_2.$$
Hence,
$$x_2^2 = 1 - y_2^2 = 1 - \tilde{y}_2^2 = \tilde{x}_2^2. \tag{2}$$
On the other hand,
$$(R + rx_2)^2 + r^2 y_2^2 = \|f(a)\|^2 = \|f(\tilde{a})\|^2 = (R + r\tilde{x}_2)^2 + r^2 y_2^2. \tag{3}$$
Using (??) and (??) we obtain
$$2rRx_2 = 2rR\tilde{x}_2,$$
hence $x_2 = \tilde{x}_2$. Finally, using $x_2 = \tilde{x}_2$ and
$$(R + rx_2)(x_1, y_1) = (R + rx_2)(\tilde{x}_1, \tilde{y}_1)$$
we conclude $(x_1, y_1) = (\tilde{x}_1, \tilde{y}_1)$. Next we prove that f is smooth. Consider a point $((x_1, y_1), (x_2, y_2)) \in \mathbb{S}^1 \times \mathbb{S}^1$. There is a local chart (U, ϕ) with $((x_1, y_1), (x_2, y_2)) \in U$ and
$$\phi^{-1} : \phi(U) \subset \mathbb{R}^2 \to \mathbb{S}^1 \times \mathbb{S}^1, \qquad \phi^{-1}(\theta, \varphi) = \big((\cos\theta, \sin\theta), (\cos\varphi, \sin\varphi)\big). \tag{4}$$

Consider the composition
$$f \circ \phi^{-1} : \phi(U) \subset \mathbb{R}^2 \to \mathbb{R}^3, \qquad f \circ \phi^{-1}(\theta, \varphi) = \big((R + r\cos\theta)\cos\varphi,\ (R + r\cos\theta)\sin\varphi,\ r\sin\theta\big), \tag{5}$$
which is clearly smooth.

(b) The inverse $f^{-1} : f(\mathbb{S}^1 \times \mathbb{S}^1) \to \mathbb{S}^1 \times \mathbb{S}^1$ is given by
$$f^{-1}(x, y, z) = \big((R + rL(x, y, z))^{-1}(x, y),\ (L(x, y, z), r^{-1}z)\big),$$
where
$$L(x, y, z) = \frac{\|(x, y, z)\|^2 - R^2 - r^2}{2Rr}.$$
This function is continuous because, for
$$(x, y, z) = f((x_1, y_1), (x_2, y_2)),$$

one has

$$|rL(x,y,z)| = \frac{\|(x,y,z)\|^2 - R^2 - r^2}{2R} = r|x_2| < R.$$

Therefore, f is a homeomorphism onto its image. Let $((x_1,y_1),(x_2,y_2)) \in \mathbb{S}^1 \times \mathbb{S}^1$ and (U,ϕ) local chart as in Eq. (??).

Usign Eq. (??) we compute the Jacobian of $f \circ \phi^{-1}$

$$J(f \circ \phi^{-1})(\theta,\varphi) = \begin{pmatrix} -r\sin\theta\cos\varphi & -(R + r\cos\theta)\sin\varphi \\ -r\sin\theta\sin\varphi & (R + r\cos\theta)\cos\varphi \\ r\cos\theta & 0 \end{pmatrix}.$$

If $r\cos\theta \neq 0$, the columns of J are linearly independent. If $r\cos\theta = 0$, then $\cos\theta = 0$ and $\sin\theta = \pm 1$, and again the columns are linearly independent. Hence $\operatorname{rank}(J) = 2$, and df_x is injective for all $x \in \mathbb{S}^1 \times \mathbb{S}^1$.

1. The of F is

$$\nabla F(x,y,z) = \big(f_1(x,y,z),\, f_2(x,y,z),\, f_3(x,y,z)\big),$$

where

$$\begin{aligned} f_1(x,y,z) &= 4x(R^2 - x^2 - y^2 - z^2 + r^2), \\ f_2(x,y,z) &= 4y(R^2 - x^2 - y^2 - z^2 + r^2), \\ f_3(x,y,z) &= -4z(x^2 + y^2 + z^2 + R^2 - r^2). \end{aligned}$$

Suppose $\nabla F(x,y,z) = 0$ with $(x,y,z) \in F^{-1}(0)$. Since $R^2 - r^2 > 0$,

$$x^2 + y^2 + z^2 + R^2 - r^2 > 0.$$

Thus $f_3(x,y,z) = 0$ implies $z = 0$. Since $F(x,y,0) = 0$,

$$4(x^2 + y^2)R^2 = (x^2 + y^2 + R^2 - r^2)^2. \tag{6}$$

If $R^2 + r^2 - x^2 - y^2 = 0$, then (??) gives

$$4(x^2 + y^2)R^2 = 4R^4,$$

hence $x^2 + y^2 = R^2$ and then $r^2 = 0$, a contradiction. Thus $R^2 + r^2 - x^2 - y^2 \neq 0$. In particular, $f_1(x,y,0) = f_2(x,y,0) = 0$ imply $x = y = 0$, which is impossible because

$$F(0,0,0) = R^2 - r^2 \neq 0.$$

We now prove

$$f(\mathbb{S}^1 \times \mathbb{S}^1) = F^{-1}(0).$$

$\subset$ For a point of the form

$$((R + rx_2)x_1, (R + rx_2)y_1, ry_2) = f((x_1,y_1),(x_2,y_2)),$$

we compute

$$\begin{aligned} &F((R + rx_2)x_1, (R + rx_2)y_1, ry_2) \\ &= 4(R + rx_2)^2R^2(x_1^2 + y_1^2) - \big((R + rx_2)^2(x_1^2 + y_1^2) + r^2y_2^2 + R^2 - r^2\big)^2 \\ &= 4R^2(R + rx_2)^2 - (2R(R + rx_2))^2 = 0. \end{aligned}$$

$\supset$ Let $(x, y, z) \in F^{-1}(0)$. Then x and y cannot both be zero, since

$$F(0,0,z) = -(z^2 + R^2 - r^2)^2 < 0.$$

Define

$$(x_1, y_1) = \frac{1}{\sqrt{x^2+y^2}}(x, y), \qquad (x_2, y_2) = \left(\frac{\sqrt{x^2+y^2} - R}{r}, \frac{z}{r} \right).$$

Since $F(x, y, z) = 0$, we get

$$x_2^2 + y_2^2 = \frac{x^2 + y^2 + z^2 + R^2 - 2R\sqrt{x^2+y^2}}{r^2} = 1.$$

Thus $(x_1, y_1), (x_2, y_2) \in \mathbb{S}^1$, and

$$f((x_1, y_1), (x_2, y_2)) = (x, y, z).$$