Exercise 1. *Consider $(X, \mathfrak{T}_X), (Y, \mathfrak{T}_Y)$ topological spaces and $p : X \to Y$ continuous function. We say that p is a quotient map if it is surjective and*

$$\mathfrak{T}_Y = \{U \in \mathcal{P}(Y) : p^{-1}(U) \in \tau_X\}.$$

- *Suppose that $p : X \to Y$ is a quotient map. Let $(Z, \mathfrak{T}_Z)$ be a topological space and let $f : X \to Z$ be a continuous function such that*

$$p(x) = p(x') \to f(x) = f(x'). \tag{1}$$

Show that there exists a continuous function $\tilde{f} : Y \to Z$ such that $\tilde{f} \circ p = f$.

- *Suppose that $p : X \to Y$ is a quotient map. Let $U \subset X$ open set such that $p^{-1}(p(U)) = U$. Let $\mathfrak{T}_U, \mathfrak{T}_{p(U)}$ the relative topologies on $U, p(U)$ respectively. Show that $p|_U : U \to p(U)$ is a quotient map, namely*

$$\{V \cap p(U) : V \in \mathfrak{T}_Y\} = \{V \subset p(U) : (p|_U)^{-1}(V) \in \mathfrak{T}_U\}.$$

Solution 1. *Let us define $\tilde{f} : Y \to Z$ by*

$$\tilde{f}(y) = f(x), \quad \text{for some } x \in X \text{ with } p(x) = y.$$

*Note that $\tilde{f}$ is well defined, because p is surjective and Eq. (**??**). Moreover, by definition $\tilde{f} \circ p = f$. Now we prove that $\tilde{f}$ is continuous. Let $V \in \tau_Z$, then since f is continuous one has*

$$\tau_X \ni f^{-1}(V) = p^{-1}(\tilde{f}^{-1}(V)).$$

Since p is a qoutient map, the last equation implies $\tilde{f}^{-1}(V) \in \tau_Y$ then $\tilde{f}$ is continuous.

$\subset$ *Suppose that $W = V \cap p(U)$ with $V \in \tau_p$. Then one has*

$$(p|_U)^{-1}(W) = p^{-1}(W) \cap U = p^{-1}(V) \cap p^{-1}(p(U)) \cap U = p^{-1}(V) \cap U \in \tau_U$$

$\supset$ *Suppose that $W \subset p(U)$ and $(p|_U)^{-1}(W) \in \tau_U$. Then $p^{-1}(W) \subset p^{-1}(p(U)) = U$, and*

$$\tau_U \ni (p|_U)^{-1}(W) = p^{-1}(W) \cap U = p^{-1}(W),$$

then $p^{-1}(W) = O \cap U$ with $O \in \tau_X$. Since $U \in \tau_X$ then $p^{-1}(W) \in \tau_X$ and since p is quotient then $W \in \tau_Y$.

Exercise 2. *Consider the following equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$:*

$$x \sim y \iff x = \lambda y, \text{ for some } \lambda \neq 0. \tag{2}$$

Denote by $\mathbb{P}^n\mathbb{R} := \{[x] : x \in \mathbb{R}^{n+1} \setminus \{0\}\}$ to the set of equivalence classes and consider on it the quoutient topology induced by the map

$$\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n\mathbb{R}, \ \pi(x) = [x],$$

$$\mathfrak{T}_{\mathbb{P}^n\mathbb{R}} = \{U \subset \mathbb{P}^n\mathbb{R} : \pi^{-1}(U) \in \mathfrak{T}_{\mathbb{R}^{n+1}}\}.$$

(a.) Consider the set

$$U_j := \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : x_j \neq 0\} \subset \mathbb{R}^{n+1} \setminus \{0\}\}.$$

Show that $\pi(U_j) \in \mathfrak{T}_{\mathbb{P}\mathbb{R}^n}$.

(b.) Consider the function

$$\phi_j : U_j \to \mathbb{R}^n,\ \phi(x) = \frac{1}{x_j}(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_{n+1}).$$

Use the Excersice 1 to prove that there exists $\tilde{\phi}_j : \pi(U_j) \to \mathbb{R}^n$ *continuous with*

$$\tilde{\phi}_j \circ \pi = \phi_j.$$

(c.) Show that the set $\{(\tilde{\phi}_j, U_j) : j = 1, \cdots, n+1\}$ *is an Atlas for* $\mathbb{P}^n\mathbb{R}$.

Conclude that $(\mathbb{P}^n\mathbb{R}, \mathfrak{T}_{\mathbb{P}^n\mathbb{R}})$ *is a differential manifold of dimension* n. $\mathbb{P}^n\mathbb{R}$ *is called the real projective space of dimension* n.

Solution 2. *Consider the open set*

$$U_j = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : x_j \neq 0\} \subset \mathbb{R}^{n+1} \setminus \{0\},$$

and note that $\pi(U_j)$ *is open. Indeed,* $\pi^{-1}(\pi(U_j)) = U_j$, *which is an open set of* $\mathbb{R}^{n+1}$. *Consider the map*

$$\phi : U_j \to \mathbb{R}^n,\ \phi(x) = \frac{1}{x_j}(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_{n+1}),$$

which is continuous. Moreover, if $x, y \in U_j$ *and* $x \sim y$ *then* $x_i = \lambda y_i$ *and one has*

$$\phi(x) = \frac{1}{\lambda y_j}(x_1, \cdots, x_{n+1}) = \phi(y).$$

Then, there exists $\tilde{\phi}_j : \pi(U_j) \to \mathbb{R}^n$ *continuous with*

$$\tilde{\phi}_j \circ \pi = \phi_j.$$

Note that we use the fact that $\pi^{-1}(\pi(U_j)) = U_j$ *which implies that* $\pi : U_j \to \pi(U_j)$ *is a quotient map. Let*

$$\psi_j : \mathbb{R}^n \to \pi(U_j),\ \psi_j(x_1, \cdots, x_n) = [x_1, \cdots, x_{j-1}, 1, x_{j+1}, \cdots, x_n].$$

Note that $\tilde{\phi}_j \circ \psi_j = I$ *and* $\psi_j \circ \tilde{\phi}_j = I$. *Then* $\tilde{\phi}_j$ *is an homeomorphism. The change of charts can be computed explicitely.*

Exercise 3. *Consider the set*

$$M := \{(x, 1), (x, -1) : x \in \mathbb{R}\} \subset \mathbb{R}^2,$$

endowed with the relative topology. Consider the following equivalence relation on M,

$$(x, y) \sim (w, z) \iff x = w \neq 0.$$

Show that $M/\sim$ *endowed with the quoutient topology satisfies the following properties.*

- *For all $x \in M/\sim$ there exist open sets $U \subset M/\sim$, with $x \in U$ and an homeomorphism $\phi : U \to V \subset \mathbb{R}$.*

- *$M/\sim$ is second numerable.*

- *$M/\sim$ is not a Haussdorf space.*

Solution 3. *Note that $U_\pm := (M/\sim) \setminus \{(0, \pm 1)\}$ is an open set. Indeed,*

$$\pi^{-1}(U_\pm) = M \setminus \{(0, \pm 1)\}$$

which is an open set of M. Let us consider $\phi : M \to \mathbb{R}$, $\phi(x, \pm 1) = x$, which is continuous and if $(x, y) \equiv (w, y)$ then $\phi(x, y) = \phi(w, y)$. Then, there exists a continuous function $\tilde{\phi} : M/\sim \to \mathbb{R}$ sucht that $\tilde{\phi} \circ \pi = \phi$. Let us define

$$\psi_\pm : \mathbb{R} \to U_\pm, \ \psi_\pm(x) = [(x, \pm 1)],$$

and note that $\psi_\pm \circ \tilde{\phi}|_{U_\pm} = I$, $\tilde{\phi}|_{U_\pm} \circ \psi_\pm = I$. Take open sets U, V with $[(x, 1)] \in U$ and $[(x, -1)] \in V$. Since U is open, $\pi^{-1}(U)$ is an open of M this and the fact that $(0, 1) \in \pi^{-1}(U)$ implies that there exists $\varepsilon > 0$ such that for all $|t| < \varepsilon$, $(t, 1) \in \pi^{-1}(U)$. Therefore,

$$\{[(t, 1)] : 0 < |t| < \varepsilon\} \subset U.$$

In the same manner for some $\varepsilon' > 0$ one has

$$\{[(t, -1)] : 0 < |t| < \varepsilon'\} \subset V.$$

Then, for $0 < |t| < \min(\varepsilon, \varepsilon')$ one has $U \ni [(t, 1)] = [(t, -1)] \in V$.

Exercise 4. 1. *Finde eine Einbettung*

$$f : S^1 \times S^1 \to \mathbb{R}^3$$

und beschreibe sie explizit.

2. *Finde eine Funktion*

$$F \in C^1(\mathbb{R}^3; \mathbb{R})$$

derart, dass $\operatorname{ran} f = F^{-1}(0)$ *und 0 ein regulärer Wert von F ist.*

Exercise 5. 1. *Gebe eine differenzierbare Abbildung*

$$f : \mathbb{R}^2 \to \mathbb{R}^3$$

an, so dass $\mathbb{R}^2/\mathbb{Z}^2$ und $\operatorname{ran} f$ *diffeomorph sind, wobei $\mathbb{Z}^2$ auf $\mathbb{R}^2$ wie in Aufgabe 14 wirkt.*

2. *Zeige, dass eine n-dimensionale Mannigfaltigkeit M, die ein Produkt von Sphären ist, eine Einbettung in $\mathbb{R}^{n+1}$ besitzt.*

a. *Show that the maping*

$$(t, x) \mapsto e^t x$$

is an embending from $\mathbb{R} \times \mathbb{S}^d$ *to* $\mathbb{R}^{d+1}$

b. *Show that* $\mathbb{S}^d \times \mathbb{R}^k$ *can be embeded in* $\mathbb{R}^{d+k}$

c. *Show that the product...*

Exercise 6. *Wir bezeichnen mit* $\mathbb{S}^2$ *die Einheitskugel in* $\mathbb{R}^3$ *und mit* P^2 *die reelle Projektivebene. Betrachten wir die folgende Abbildung:*

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \quad (x, y, z) \mapsto (2xz,\, 2yz,\, 1 - 2z^2).$$

1. *Zeigen Sie, dass die Einschränkung von* f *auf* $\mathbb{S}^2$ *eine* C^∞*-Abbildung von* $\mathbb{S}^2$ *nach* $\mathbb{S}^2$ *ist.*

2. *Bestimmen Sie die kritischen Werte von* $f|_{\mathbb{S}^2}$.

3. *Zeigen Sie, dass eine glatte Abbildung*

$$h : P^2 \to \mathbb{S}^2$$

existiert, sodass

$$f|_{\mathbb{S}^2} = h \circ \pi,$$

wobei $\pi : \mathbb{S}^2 \to P^2$ *die kanonische Projektion ist.*

4. *Zeigen Sie, dass* h *surjektiv ist, und bestimmen Sie die kritischen und regulären Werte von* h.