I. Mathematical Prerequisites

I.1. Analysis in one and several Real Variables

We list a few topics from analysis in one and in several real variables that we assume the reader to be familiar with:

- Real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, and their d-fold cartesian products $\mathbb{R}^d$ and $\mathbb{C}^d$, $d \in \mathbb{N}$.
- Real and complex sequences, series, their convergence, and criteria to decide for convergence or divergence.
- Basic topological notions such as inner points, accumulation points, open sets, closed sets, compact sets in $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^d$, and $\mathbb{C}^d$.
- Continuity of maps and its various characterizations.
- Differentiability and basic rules of differentiation, such as Leibniz rule and the chain rule.
- (Riemann) Integration, integration by parts, the fundamental theorem of calculus.
- Partial derivatives, gradient, and Jacobi matrix.
- Local extrema, local extrema under constraints, method of Lagrange multipliers.
- Integration of several variables.
- Basic inequalities: Cauchy-Schwarz, Hölder, Minkowski.

I.2. Introductory Linear Algebra

We also list a few topics from linear algebra that we assume the reader to be familiar with:

- Real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, and their d-fold cartesian products $\mathbb{R}^d$ and $\mathbb{C}^d$, $d \in \mathbb{N}$.

- Vector spaces, subspaces, linear span and their generating sets.
- Linear dependence, linear independence, basis, and dimension.
- Linear maps and their matrix representations.
- Matrices, matrix product, deteminants, equivalence of invertibility of a matrix to the nonvanishing of its determinant.
- Eigenvalues and eigenvectors, diagonalizability.

I.3. Norms and Scalar Products

I.3.1. Banach Spaces

In this section we define Banach spaces and collect some of their basic properties. We recall that $\mathbb{K}$ denotes the field $\mathbb{R}$ of real numbers or the field $\mathbb{R}$ of complex numbers. Statements made involving $\mathbb{K}$ hold for both $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

Definition I.1. Let X be a $\mathbb{K}$-vector space. A map $\| \cdot \| : X \to \mathbb{R}_0^+$ is called **Norm (on X)** :⇔

(i)

$$\forall\, x \in X : \qquad \big\{ \|x\| = 0 \;\Leftrightarrow\; x = 0 \big\} \tag{I.1}$$

(ii)

$$\forall\, x \in X,\ \lambda \in \mathbb{K} : \qquad \|\lambda x\| \;=\; |\lambda| \cdot \|x\|, \tag{I.2}$$

(iii)

$$\forall\, x, y \in X : \qquad \|x + y\| \;\leq\; \|x\| + \|y\|. \tag{I.3}$$

In this case $(X, \| \cdot \|)$ is said to be a **normed (vector) space**. We denote by

$$B_X(x, r) \;:=\; \big\{ y \in X \;\big|\, \|x - y\| < r \big\} \tag{I.4}$$

the **open ball about $x \in X$ of radius $r > 0$**.

Definition I.2. Let $(X, \| \cdot \|)$ be a normed vector space over $\mathbb{K}$.

(i) A sequence $(x_n)_{n=1}^\infty \in X^{\mathbb{N}}$ is **convergent** :⇔

$$\exists x \in X\ \forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 : \quad \|x_n - x\| \leq \varepsilon. \tag{I.5}$$

(ii) A sequence $(x_n)_{n=1}^\infty \in X^{\mathbb{N}}$ is called **Cauchy sequence** :⇔

$$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall m > n \geq n_0 : \quad \|x_m - x_n\| \leq \varepsilon. \tag{I.6}$$

(iii) If every Cauchy sequence in X is convergent, $(X, \|\cdot\|)$ is said to be **complete**, and we call $(X, \|\cdot\|)$ a **Banach space**.

(iv) A subset $S \subseteq X$ is **dense**, if $\overline{S} = X$ or, equivalently, if

$$\forall\, x \in X\,,\ \varepsilon > 0\ \exists\, y_\varepsilon \in S: \qquad \|x - y_\varepsilon\| \ \leq\ \varepsilon\,. \tag{I.7}$$

Remarks and Examples. We first list a few examples of Banach spaces.

- For $d \in \mathbb{N}$ the $\mathbb{K}$-vector space $\big(\mathbb{K}^d, \|\cdot\|_2\big)$ is a Banach space with respect to the euclidean/unitary norm $\|x\|_2 := \langle x|x\rangle^{1/2}$, with $\langle x|y\rangle := \sum_{\nu=1}^d x_\nu y_\nu$ ($\mathbb{K} = \mathbb{R}$) or $\langle x|y\rangle := \sum_{\nu=1}^d \overline{x}_\nu y_\nu$ ($\mathbb{K} = \mathbb{C}$).
- For $d \in \mathbb{N}$ and $1 \leq p < \infty$, the vector space $\big(\mathbb{K}^d, \|\cdot\|_p\big)$ is a $\mathbb{K}$-Banach space with respect to the p-norm $\|x\|_p := \big(|x_1|^p + \ldots + |x_d|^p\big)^{1/p}$. The triangle inequality $\|x+y\|_p \leq \|x\|_p + \|y\|_p$ is the classical Minkowski inequality in analysis.
- For $d \in \mathbb{N}$, the vector space $\big(\mathbb{K}^d, \|\cdot\|_\infty\big)$ is a $\mathbb{K}$-Banach space with respect to the supremum norm or ∞-norm $\|x\|_\infty := \max\big(|x_1|, \ldots, |x_d|\big)$. This corresponds to the case $p = \infty$.
- Recall that a **subspace** of a $\mathbb{K}$-vector space X is a subset $Z \subseteq X$ which itself is a $\mathbb{K}$-vector space. If $\big(X, \|\cdot\|_X\big)$ is a Banach space and $Z \subseteq X$ is a subspace then $\big(Z, \|\cdot\|_X\big)$ is itself a Banach space if, and only if, it is a closed subset of X.

I.3.2. Linear Operators

Definition I.3. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two $\mathbb{K}$-Banach spaces. Wir denote by

$$\mathcal{B}(X;Y) \ :=\ \Big\{ A : X \to Y \ \Big|\ A \text{ is linear, } \|A\|_{\mathcal{B}(X;Y)} < \infty \Big\} \tag{I.8}$$

the space of **bounded (linear) operators (from X to Y)**, where

$$\|A\|_{\mathcal{B}(X;Y)} \ :=\ \sup_{x \in X \setminus \{0\}} \left\{ \frac{\|Ax\|_Y}{\|x\|_X} \right\} \ =\ \sup_{x \in X, \|x\|_X = 1} \big\{ \|Ax\|_Y \big\} \tag{I.9}$$

is the **operator norm of A**. If no confusion is possible, we frequently write $\|\cdot\|_{\mathcal{B}(X;Y)} =: \|\cdot\|_{\mathrm{op}}$.

Remarks and Examples.

- $\big(\mathcal{B}(X;Y), \|\cdot\|_{\mathcal{B}(X;Y)}\big)$ is itself a $\mathbb{K}$-Banach space, where the vector space structure is defined by $(A + \lambda B)(x) := A(x) + \lambda B(x)$.
- If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two Banach spaces over $\mathbb{K}$ and $A : X \to Y$ is linear, then the following two statements are equivalent:

$$\big\{ \|A\|_{\mathcal{B}(X;Y)} \ <\ \infty \big\} \qquad \Leftrightarrow \qquad \big\{ A:\ X \to Y \text{ is continuous} \big\}. \tag{I.10}$$

I.3.3. Hilbert Spaces

In this section we define (complex) Hilbert spaces and collect some basic facts about these and about the spectral theory of bounded operators on Hilbert spaces.

Definition I.4.

(i) Let X be an $\mathbb{C}$-Vector space. A map $\langle\cdot|\cdot\rangle : X \times X \to \mathbb{R}$ is called **sesquilinear form (on X)**

$$:\Leftrightarrow \forall\, \alpha, \beta \in \mathbb{R}\, \forall\, x, y, w, z \in X : \quad \langle \alpha x + y \mid \beta w + z \rangle \;=\; \overline{\alpha}\beta\langle x|w\rangle + \overline{\alpha}\langle x|z\rangle + \beta\langle y|z\rangle + \langle y|z\rangle. \tag{I.11}$$

If furthermore

$$:\Leftrightarrow \forall\, x, y \in X : \quad \langle x|y\rangle \;=\; \overline{\langle y|x\rangle}, \tag{I.12}$$

holds true, the sesquilinear form $\langle\cdot|\cdot\rangle$ is called **symmetric**.

(ii) A symmetric sesquilinear form $\langle\cdot|\cdot\rangle : X \times X \to \mathbb{C}$ is called **positiv definite**

$$:\Leftrightarrow \quad \forall\, x \in X \setminus \{0\} : \quad \langle x|x\rangle \;>\; 0. \tag{I.13}$$

If only $\langle x|x\rangle \geq 0$ holds true, for all $x \in X$ (but possibly there are $x \neq 0$ with $\langle x|x\rangle = 0$), then $\langle\cdot|\cdot\rangle : X \times X \to \mathbb{C}$ is called **positiv semidefinite**.

(iii) If X is a $\mathbb{C}$-vector space and $\langle\cdot|\cdot\rangle : X \times X \to \mathbb{C}$ is a symmetric, positive definite sesquilinear form on X, then $\langle\cdot|\cdot\rangle$ is called **scalar product (X)** and $\big(X, \langle\cdot|\cdot\rangle\big)$ is called a **pre-Hilbert space**.

Theorem I.5 (Cauchy-Schwarz Inequality)**.** *If X is a $\mathbb{C}$-vector space equipped with a symmetric, positiv semidefinite sesquilinear form $\langle\cdot|\cdot\rangle : X \times X \to \mathbb{C}$ then*

$$\forall x, y \in X : \quad \big|\langle x|y\rangle\big| \;\leq\; \sqrt{\langle x|x\rangle}\, \sqrt{\langle y|y\rangle}. \tag{I.14}$$

Corollary I.6. *If $\big(\mathcal{H}, \langle\cdot|\cdot\rangle\big)$ is a $\mathbb{C}$-pre-Hilbert space then*

$$\|\cdot\| : \mathcal{H} \;\to\; \mathbb{R}_0^+, \quad \|x\| \;:=\; \langle x|x\rangle^{1/2} \tag{I.15}$$

*defines a norm on $\mathcal{H}$, the **norm induced by $\langle\cdot|\cdot\rangle$**.*

Definition I.7. If a pre-Hilbert space $\big(\mathcal{H}, \langle\cdot|\cdot\rangle\big)$ over $\mathbb{C}$ is complete with respect to the norm induced by $\langle\cdot|\cdot\rangle$, then we call $\big(\mathcal{H}, \langle\cdot|\cdot\rangle\big)$ a **Hilbert space**.

Remarks and Examples.

- For $d \in \mathbb{N}$, the space $\big(\mathbb{C}^d, \langle \cdot | \cdot \rangle_{\text{unit}}\big)$ is a (complex) Hilbert space, where

$$\forall\, \vec{x} = (x_1, \ldots, x_d)^t\,,\ \vec{y} = (y_1, \ldots, y_d)^t \in \mathbb{C}^d : \quad \big\langle \vec{x} \big| \vec{y} \big\rangle_{\text{unit}} \ := \ \sum_{\nu=1}^{d} \overline{x_\nu}\, y_\nu \tag{I.16}$$

  defines the **unitary scalar product**.

- Let $(\Omega, \mathfrak{A}, \mu)$ be a measure space. Then $\big(L^2(\Omega; \mathbb{C}), \langle \cdot | \cdot \rangle\big)$ is a Hilbert space with scalar product

$$\forall f, g \in L^2(\Omega; \mathbb{C}) : \qquad \langle f | g \rangle \ := \ \int_\Omega \overline{f(\omega)}\, g(\omega)\, d\mu(\omega). \tag{I.17}$$

Definition I.8. Suppose that $\big(\mathcal{H}, \langle \cdot | \cdot \rangle\big)$ is a Hilbert space.

(i) Two vectors $x, y \in \mathcal{H}$ are **orthogonal, $\boldsymbol{x \perp y}$** $:\Leftrightarrow \langle x | y \rangle = 0$.

(ii) If $\mathcal{A} \subseteq \mathcal{H}$ is a subset then

$$\mathcal{A}^\perp \ := \ \big\{ x \in \mathcal{H} \,\big|\, \forall a \in \mathcal{A} : \ \langle a | x \rangle = 0 \big\} \tag{I.18}$$

is the **orthogonal complement to $\boldsymbol{\mathcal{A}}$**.

(iii) A subset $\mathcal{B} \subseteq \mathcal{H}$ is called **orthonormal** $:\Leftrightarrow$

$$\forall x, y \in \mathcal{B},\ x \neq y : \qquad \langle x | x \rangle = \langle y | y \rangle = 1, \quad \langle x | y \rangle = 0. \tag{I.19}$$

(iv) A subset $\mathcal{E} \subseteq \mathcal{H}$ is called an **orthonormal basis (ONB) (of $\boldsymbol{\mathcal{H}}$)**

$$\begin{aligned} :\Leftrightarrow \quad & \mathcal{E} \text{ is orthonormal} \quad \text{and} \quad \overline{\operatorname{span}(\mathcal{E})} = \mathcal{H} \\ \Leftrightarrow \quad & \mathcal{E} \text{ is orthonormal} \quad \text{and} \quad \text{for any given } x \in \mathcal{H} \text{ and } \varepsilon > 0 \\ & \exists\, e_1, \ldots, e_N \in \mathcal{E},\ \alpha_1, \ldots, \alpha_N \in \mathbb{C} : \ \Big\| x - \big(\alpha_1 e_1 + \ldots + \alpha_N e_N\big) \Big\| \ \leq \ \varepsilon\,. \end{aligned} \tag{I.20}$$

Remarks and Examples.

- For any subset $\mathcal{A} \subseteq \mathcal{H}$ its orthogonal complement $\mathcal{A}^\perp$ is a closed subspace.
- For any subset $\mathcal{A} \subseteq \mathcal{H}$, we have that $\mathcal{A} \cap \mathcal{A}^\perp \subseteq \{0\}$ is a closed subspace.
- If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{H}$ then $\mathcal{A}^\perp \supseteq \mathcal{B}^\perp$.
- If a subset $\mathcal{A} \subseteq \mathcal{H}$ is orthonormal then it is linearly independent.
- The canonical basis $\{e_1, \ldots, e_d\} \subseteq \mathbb{C}^d$, with

$$e_1 \ = \ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 \ = \ \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots \quad e_d \ = \ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \tag{I.21}$$

  is an ONB in $\mathbb{C}^d$ with respect to the euclidean or unitary scalar product, respectively.

Theorem I.9 (Gram-Schmidt Orthonormalization Procedure)**.** *Suppose that* $\big(\mathcal{H}, \langle\cdot|\cdot\rangle\big)$ *is a separable Hilbert space. Then there exists a countable ONB* $\{e_n\}_{n=1}^L \subseteq \mathcal{H}$*, where* $L = \dim_{\mathbb{C}}(\mathcal{H}) \in \mathbb{N} \cup \{\infty\}$ *denotes the dimension of* $\mathcal{H}$*. If* $\{e_n\}_{n=1}^L \subseteq \mathcal{H}$ *is such an ONB, then*

$$\forall x \in \mathcal{H}: \qquad x \;=\; \sum_{n=1}^{L} \langle e_n | x\rangle \, e_n \,. \tag{I.22}$$

Remarks and Examples.

- Eq. (I.22) says that the coefficients in the approximation (I.20) can be computed by taking scalar products $\alpha_n = \langle e_n | x\rangle$.
- For $\varphi, \psi \in \mathcal{H}$ we use **Dirac's ket-bra notation** and define $|\varphi\rangle\langle\psi| \in \mathcal{B}(\mathcal{H})$ by
$$\forall x \in \mathcal{H}: \qquad |\varphi\rangle\langle\psi|(x) \;:=\; \langle\psi|x\rangle \, \varphi \,. \tag{I.23}$$
- Note that, given $\varphi, \psi \in \mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$, we have for all $x \in \mathcal{H}$ that
$$\big[A \circ |\varphi\rangle\langle\psi|\big](x) \;=\; A\big[\langle\psi|x\rangle\, \varphi\big] \;=\; \langle\psi|x\rangle\, A(\varphi) \;=\; |A\varphi\rangle\langle\psi|(x) \tag{I.24}$$
and
$$\Big\langle y \Big| |\varphi\rangle\langle\psi|(x) \Big\rangle \;=\; \Big\langle y \Big| \langle\psi|x\rangle\, \varphi \Big\rangle \;=\; \langle\psi|x\rangle\, \langle y|\varphi\rangle \;=\; \Big\langle \langle\varphi|y\rangle\, \psi \Big| x \Big\rangle \;=\; \Big\langle |\psi\rangle\langle\varphi|(y) \Big| x \Big\rangle \,. \tag{I.25}$$
Eqs. (I.24) and (I.25) imply that
$$A \circ |\varphi\rangle\langle\psi| \;=\; |A\varphi\rangle\langle\psi| \,, \;\; \big(|\varphi\rangle\langle\psi|\big)^* \;=\; |\psi\rangle\langle\varphi| \,, \;\; \text{and} \;\; |\varphi\rangle\langle\psi| \circ A \;=\; |\varphi\rangle\langle A^*\psi| \,. \tag{I.26}$$
- If $L \in \mathbb{N} \cup \{\infty\}$ and $\{\varphi_1, \ldots, \varphi_L\} \subseteq \mathcal{H}$ is an ONB then (I.22) can be written by means of Dirac's ket-bra notation as a decomposition of the identity,
$$\mathbf{1}_{\mathcal{H}} \;=\; \sum_{n=1}^{L} |\varphi_n\rangle\langle\varphi_n| \,. \tag{I.27}$$
- If $N < \infty$ and $\{m_1, \ldots, m_N\} \subseteq \mathcal{M}$ is an ONB then the orthogonal projection $P \in \mathcal{B}(\mathcal{H})$ onto $\mathcal{M}$ is given by
$$P \;=\; \sum_{n=1}^{N} |m_n\rangle\langle m_n| \,. \tag{I.28}$$

Theorem I.10 (Riesz Representation Theorem)**.** *Let* $\big(\mathcal{H}, \langle\cdot|\cdot\rangle\big)$ *be a Hilbert space and* $\mathcal{H}^* := \mathcal{B}(\mathcal{H}; \mathbb{C})$ *its dual space (see Definition I.11). Then, for every bounded linear functional* $\ell \in \mathcal{H}^*$*, there exists exactly one vector* $y_\ell \in \mathcal{H}$*, such that* $\ell = \langle y_\ell|\cdot\rangle$*, i.e.*

$$\forall x \in \mathcal{H}: \qquad \ell(x) \;=\; \langle y_\ell|x\rangle. \tag{I.29}$$

Furthermore $\|\ell\|_{\mathcal{H}^*} = \|y_\ell\|_{\mathcal{H}}$ *and*

$$\forall x \in \mathcal{H}: \qquad \|x\| \;=\; \sup\Big\{ |\langle y|x\rangle| \;\Big|\; y \in \mathcal{H}\,,\; \|y\| = 1 \Big\}. \tag{I.30}$$

Proof. We only give a proof of Eq. (I.30) because the argument is instructive. Note that (I.30) holds trivially true for $x = 0$, and we may assume that $x \neq 0$. We abbreviate the right side of (I.30) by

$$N(x) \ := \ \sup \left\{ |\langle y|x\rangle| \ \middle| \ y \in \mathcal{H} \, , \ \|y\| = 1 \right\} . \tag{I.31}$$

If $y \in \mathcal{H}$ with $\|y\| = 1$ then the Cauchy-Schwarz inequality implies that

$$|\langle y|x\rangle| \ \leq \ \|x\| \cdot \|y\| \ = \ \|x\| \, . \tag{I.32}$$

Taking the supremum over all $y \in \mathcal{H}$ with $\|y\| = 1$, we obtain $N(x) \leq \|x\|$. Next we define $\hat{x} := \frac{1}{\|x\|} x \in \mathcal{H}$, which is possible thanks to $x \neq 0$, and observe that $\|\hat{x}\| = 1$. Hence,

$$\|x\| \ = \frac{\|x\|^2}{\|x\|} \ = \ \frac{\langle x|x\rangle}{\|x\|} \ = \ \langle \hat{x}|x\rangle \ \leq \ |\langle \hat{x}|x\rangle| \ \leq \ N(x) \, , \tag{I.33}$$

which yields $\|x\| \leq N(x)$. Hence $\|x\| = N(x)$. $\square$

I.4. SUPPLEMENTARY MATERIAL

I.4.1. Banach Spaces

Remarks and Examples.

- For a normed space $(X, \| \cdot \|)$
- Let $(X, \| \cdot \|)$ be a normed space. The **norm topology** is the topology generated by the system of open balls in X.
- Two norms $\| \cdot \|_1, \| \cdot \|_2 : X \to \mathbb{R}_0^+$ on a $\mathbb{K}$-vector space X are called **equivalent**, if there exists a constant $c > 0$, such that

  $$\forall \, x \in X : \qquad c \, \|x\|_1 \ \leq \ \|x\|_2 \ \leq \ c^{-1} \, \|x\|_1 \tag{I.34}$$

  holds true. In this case $\| \cdot \|_1$ and $\| \cdot \|_2$ induce the same topology on X.
- If $(X, \| \cdot \|)$ is a Banach space, which contains a countable dense set, then $(X, \| \cdot \|)$ is called **separable**.

Remarks and Examples. We illustrate the notion of separablity on examples.

- The subset of vectors $(x_1, \ldots, x_d) \in \mathbb{K}^d$ with rational coefficients $x_\nu \in \mathbb{K}_\mathbb{Q}$ is countable and dense in $\mathbb{K}^d$. Here, $\mathbb{K}_\mathbb{Q} := \mathbb{Q}$, if $\mathbb{K} = \mathbb{R}$, and $\mathbb{K}_\mathbb{Q} := \mathbb{Q} + i\mathbb{Q}$, if $\mathbb{K} = \mathbb{C}$. Therefore, $\big(\mathbb{K}^d, \| \cdot \|_2\big)$ is separable.

Next, for $1 \le p < \infty$, we define

$$\ell^p(\mathbb{N}) := \left\{ \underline{x} \in \mathbb{K}^{\mathbb{N}} \;\middle|\; \|\underline{x}\|_p := \Big(\sum_{\nu=1}^{\infty} |x_\nu|^p \Big)^{1/p} < \infty \right\}, \tag{I.35}$$

$$\ell^\infty(\mathbb{N}) := \left\{ \underline{x} \in \mathbb{K}^{\mathbb{N}} \;\middle|\; \|\underline{x}\|_\infty := \sup_{\nu \in \mathbb{N}} |x_\nu| < \infty \right\}. \tag{I.36}$$

Then $\big(\ell^p(\mathbb{N}), \|\cdot\|_p\big)$ is a $\mathbb{K}$-Banach space, for $1 \le p \le \infty$. We further define

$$c_{\mathrm{fin}}^{\mathbb{Q}}(\mathbb{N}) := \left\{ \underline{x} \in \mathbb{K}_{\mathbb{Q}}^{\mathbb{N}} \;\middle|\; \exists \nu_0 \in \mathbb{N}\, \forall \nu \ge \nu_0 : \quad x_\nu = 0 \right\}, \tag{I.37}$$

$$c_{\mathrm{fin}}(\mathbb{N}) := \left\{ \underline{x} \in \mathbb{K}^{\mathbb{N}} \;\middle|\; \exists \nu_0 \in \mathbb{N}\, \forall \nu \ge \nu_0 : \quad x_\nu = 0 \right\}, \tag{I.38}$$

$$c_0(\mathbb{N}) := \left\{ \underline{x} \in \mathbb{K}^{\mathbb{N}} \;\middle|\; \lim_{\nu \to \infty} x_\nu = 0 \right\}, \tag{I.39}$$

and observe that

$$c_{\mathrm{fin}}^{\mathbb{Q}}(\mathbb{N}) \subseteq c_{\mathrm{fin}}(\mathbb{N}) \subseteq \ell^p(\mathbb{N}) \subseteq \ell^{\tilde{p}}(\mathbb{N}) \subseteq c_0(\mathbb{N}) \subseteq \ell^\infty(\mathbb{N}), \tag{I.40}$$

where $1 \le p < \tilde{p} < \infty$.

- Both $\big(c_0(\mathbb{N}), \|\cdot\|_\infty\big)$ and $\big(\ell_\infty(\mathbb{N}), \|\cdot\|_\infty\big)$ are Banach spaces with respect to the supremum norm $\|\cdot\|_\infty$. Note that $c_0(\mathbb{N}) \subset \ell_\infty(\mathbb{N})$ is a strict inclusion because, e.g., $(1, 1, 1, \ldots) \in \ell_\infty(\mathbb{N}) \setminus c_0(\mathbb{N})$.
- The set $c_{\mathrm{fin}}^{\mathbb{Q}}(\mathbb{N}) = \bigcup_{d=1}^{\infty} \mathbb{K}_{\mathbb{Q}}^d$ is a countable union of countable sets and, hence, countable itself.
- The countable subset $c_{\mathrm{fin}}^{\mathbb{Q}}(\mathbb{N}) \subseteq c_{\mathrm{fin}}(\mathbb{N})$ is dense in $c_{\mathrm{fin}}(\mathbb{N})$ with respect to the supremum norm $\|\cdot\|_\infty$.
- If $\underline{x} \in c_0(\mathbb{N})$ and $\varepsilon > 0$, there exists an $N \in \mathbb{N}$, such that $|x_\nu| \le \varepsilon$, for all $\nu > N$. Moreover, for all $\nu \in \mathbb{N}$ and $x_\nu \in \mathbb{K}$, we can find a $\tilde{x}_\nu \in \mathbb{K}_{\mathbb{Q}}$ with $|x_\nu - \tilde{x}_\nu| \le \varepsilon$. Defining $\underline{\tilde{x}} := (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N, 0, 0, \ldots)$, we hence obtain $\underline{\tilde{x}} \in c_{\mathrm{fin}}^{\mathbb{Q}}(\mathbb{N})$ and $\|\underline{x} - \underline{\tilde{x}}\|_\infty \le \varepsilon$. In other words, $c_{\mathrm{fin}}^{\mathbb{Q}}(\mathbb{N}) \subseteq c_0(\mathbb{N})$ is dense, and the Banach space $\big(c_0(\mathbb{N}), \|\cdot\|_\infty\big)$ is separable.
- If $\underline{x} \in \ell^p(\mathbb{N})$ and $\varepsilon > 0$, then there exists an $N \in \mathbb{N}$, such that $\sum_{\nu=N}^{\infty} |x_\nu|^p \le \varepsilon^p / 2^p$. Moreover, to every $\nu \in \mathbb{N}$ and $x_\nu \in \mathbb{K}$ we can find an $\tilde{x}_\nu \in \mathbb{K}_{\mathbb{Q}}$, such that $|x_\nu - \tilde{x}_\nu| \le \varepsilon / 2N$. Setting $\underline{\tilde{x}} := (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{\nu_0}, 0, 0, \ldots)$, we observe that $\underline{\tilde{x}} \in c_{\mathrm{fin}}(\mathbb{N})$ and

$$\|\underline{x} - \underline{\tilde{x}}\|_p \le \left[\sum_{\nu=1}^{N} \Big(\frac{\varepsilon}{2N} \Big)^p \right]^{1/p} + \frac{\varepsilon}{2} = \frac{\varepsilon}{2N^{(p-1)/p}} + \frac{\varepsilon}{2} \le \varepsilon. \tag{I.41}$$

  It follows that $c_{\mathrm{fin}}^{\mathbb{Q}}(\mathbb{N}) \subseteq \ell^p(\mathbb{N})$ is dense, and that the Banach space $\big(\ell^p(\mathbb{N}), \|\cdot\|_p\big)$ is separable.

- Let $\{\underline{x}^{(k)}\}_{k=1}^{\infty} \subseteq \ell^\infty(\mathbb{N})$ be a countable subset, where $\underline{x}^{(k)} = \big(x_\nu^{(k)}\big)_{\nu=1}^{\infty}$. We define $\underline{\tilde{x}} = (\tilde{x}_\nu)_{\nu=1}^{\infty} \in \ell^\infty(\mathbb{N})$ by

$$\tilde{x}_\nu := \begin{cases} 2, & \text{if } |x_\nu^{(\nu)}| \leq 1, \\ 0, & \text{if } |x_\nu^{(\nu)}| > 1. \end{cases} \tag{I.42}$$

  Then, for all $k \in \mathbb{N}$, we have that

$$\big\|\underline{\tilde{x}} - \underline{x}^{(k)}\big\|_\infty \geq \big|\tilde{x}_k - x_k^{(k)}\big| \geq 1. \tag{I.43}$$

  Consequently, $\{\underline{x}^{(k)}\}_{k=1}^{\infty} \subseteq \ell^\infty(\mathbb{N})$ is not dense. Since $\{\underline{x}^{(k)}\}_{k=1}^{\infty}$ is an arbitrary countable subset, $\ell^\infty(\mathbb{N})$ cannot be separable.

I.4.2. Linear Operators

Definition I.11. Let $(X, \|\cdot\|)$ be a Banach space over $\mathbb{K}$. The $\mathbb{K}$-Banach space $\mathcal{B}(X;\mathbb{K})$ of the bounded linear operators from X to $\mathbb{K}$ is called **dual space $X^* := \mathcal{B}(X;\mathbb{K})$ (of X)**. Elements $y^* \in X^*$ of X^* are called **bounded linear functionals** or **continuous linear functionals** [according to (I.10)]. For the value $y^*(x)$ of $y^* \in X^*$ at the point $x \in X$ we write

$$x^*(x) =: \big\langle x^*,\, x\big\rangle. \tag{I.44}$$

Remarks and Examples.

- If $y^* \in X^*$ then

$$\|y^*\|_{X^*} = \sup_{x \in X \setminus \{0\}} \left\{ \frac{|\langle y^*,\, x\rangle|}{\|x\|_X} \right\}. \tag{I.45}$$

- For $X = \mathbb{K}^d$, with $d \in \mathbb{N}$ and norm $\|\cdot\|$, the dual space X^* is isomorph zu X, that is $X^* = \mathbb{K}^d$. More specifically,
- If $\{x_1, \ldots, x_d\} \subseteq X$ is a basis then there exists a unique basis $\{x_1^*, \ldots, x_d^*\} \subseteq X^*$, such that

$$\forall\, 1 \leq i, j \leq d: \qquad \langle x_j^*,\, x_i\rangle = \delta_{ij}. \tag{I.46}$$

  This fact motivates the notation (I.44) $x^*(x) =: \big\langle x^*,\, x\big\rangle$.
- If $(X, \|\cdot\|)$ is a $\mathbb{K}$-Banach space and $Y \subseteq X$ is a closed subspace (and hence a Banach space itself), then $X^* \subseteq Y^*$. Namely, if $x^* \in X^*$ then $x^* \upharpoonright_Y \in Y^*$.
- A $\mathbb{K}$-Banach space $\big(X, \|\cdot\|\big)$ is **reflexive**, if the dual space of its dual space is X itself, $\big((X, \|\cdot\|)^*\big)^* = (X, \|\cdot\|)$.
- For $d \in \mathbb{N}$, the Banach spaces $\big(\mathbb{K}^d, \|\cdot\|_p\big)$ are reflexive, for all $1 \leq p \leq \infty$, since $\big(\mathbb{K}^d, \|\cdot\|_p\big)^* = \big(\mathbb{K}^d, \|\cdot\|_q\big)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

- For $1 < p < \infty$, the Banach spaces $\big(\ell^p(\mathbb{N}), \|\cdot\|_p\big)$ are reflexive, with $\big(\ell^p(\mathbb{N}), \|\cdot\|_p\big)^* = \big(\ell^q(\mathbb{N}), \|\cdot\|_q\big)$ with $\frac{1}{p} + \frac{1}{q} = 1$. This is essentially a consequence of Hölder's inequality,

$$\Big| \sum_{\nu=1}^{\infty} \overline{x_\nu}\, y_\nu \Big| \;\le\; \Big(\sum_{\nu=1}^{\infty} |x_\nu|^p \Big)^{1/p} \Big(\sum_{\nu=1}^{\infty} |y_\nu|^q \Big)^{1/q} . \tag{I.47}$$

I.4.3. Hilbert Spaces

Lemma I.12. *Suppose that $\big(\mathcal{H}, \langle\cdot|\cdot\rangle\big)$ is a Hilbert space, $\mathcal{M} \subseteq \mathcal{H}$ is a closed subspace, and $x \in \mathcal{H}$. Then there is a unique vector $z \in \mathcal{M}$, such that*

$$\|x - z\| \;=\; \inf_{y \in \mathcal{M}} \|x - y\|. \tag{I.48}$$

Theorem I.13. *If $\big(\mathcal{H}, \langle\cdot|\cdot\rangle\big)$ is a Hilbert space and $\mathcal{M} \subseteq \mathcal{H}$ is a closed subspace, then $\mathcal{M}$ is complementable, namely,*

$$\mathcal{H} \;=\; \mathcal{M} \,\oplus\, \mathcal{M}^\perp. \tag{I.49}$$

Corollary I.14. *Suppose that $\big(\mathcal{H}, \langle\cdot|\cdot\rangle\big)$ is a Hilbert space and $\mathcal{M} \subseteq \mathcal{H}$ a closed subspace. Define $P : \mathcal{H} \to \mathcal{H}$ by $P(x) \in \mathcal{M}$ and $\|x - P(x)\| = \inf_{y \in \mathcal{M}} \|x - y\|$. Then $P = P^2 \in \mathcal{B}(\mathcal{H})$ is an idempotent bounded linear operator on $\mathcal{H}$ with $\mathrm{Ran} P = \mathcal{M}$ and $\|P\|_{\mathrm{op}} = 1$, provided $\mathcal{M} \neq \{0\}$, and $P = 0$ otherwise. The operator P is called* ***orthogonal projection onto $\mathcal{M}$****.*