Homework Problem Set 6 for the Lecture
Introduction to Quantum Information Theory

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- problem sheet uploaded on 03-Jul-2025.
- admissible format of homework is a scan of a handwritten document converted to PDF,
- submission of homework by e-mail to v.bach@tu-bs.de until 15-Jul-2025,
- discussion of the solution in the tutorial on 18-Jul-2025.

Problem 6.1 (12 Points): Let $d \in \mathbb{N}, \mathcal{H} = \mathbb{C}^d$ and let $\mathcal{SA}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) | A = A^*\}$ be the set of self-adjoint operators on $\mathcal{H}$. Recall that for $A, B \in \mathcal{SA}$, we write $A \leq B$ if $B - A \geq 0$ is a positive operator.

(a) Prove that "$\leq$" defines a partial order on $\mathcal{SA}$, i.e., that for all $A, B, C \in \mathcal{SA}(\mathcal{H})$,

$$A \leq A\,, \qquad \text{(reflexivity)}$$
$$\{A \leq B \wedge B \leq A\} \Rightarrow A = B\,, \qquad \text{(antisymmetry)}$$
$$\{A \leq B \wedge B \leq C\} \Rightarrow A \leq C\,. \qquad \text{(transitivity)}$$

(b) Let $A, B \in \mathcal{SA}(\mathcal{H})$ be two positive operators and define

$$D := \frac{1}{2}(A + B) + \frac{1}{2}|A - B|\,. \tag{1}$$

Show that both $D \geq A$ and $D \geq B$ hold true.

(c) Show that D, as defined in (1), does not define an operator-valued supremum of A and B, in general. That is, show that $(E \geq A) \wedge (E \geq B)$ does not imply $E \geq D$.

To this end, let $\mathcal{H} = \mathbb{C}^2$ and

$$A := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

Compute D and construct $E \in \mathcal{SA}(\mathbb{C}^2)$ such that $E \geq A$ and $E \geq B$, but $E - D$ is indefinite.

Solution.

(a) Reflexivity: For all $\psi \in \mathcal{H}$, we have $\langle \psi | A\psi \rangle \leq \langle \psi | A\psi \rangle$ which is equivalent to $A \leq A$.

Antisymmetry: Let $A \geq B$ and $A \leq B$. Then $\langle \psi | A\psi \rangle \geq \langle \psi | B\psi \rangle$ and $\langle \psi | A\psi \rangle \leq \langle \psi | B\psi \rangle$, for all $\psi \in \mathcal{H}$, which is equivalent to $\langle \psi | (A - B)\psi \rangle = 0$, for all $\psi \in \mathcal{H}$. As a self-adjoint operator is completely determined by diagonal matrix elements, this implies $A - B = 0$.

Antisymmetry: Let $A \leq B$ and $B \leq C$. Then $\langle \psi | A\psi \rangle \leq \langle \psi | B\psi \rangle \leq \langle \psi | C\psi \rangle$, for all $\psi \in \mathcal{H}$. Ignoring the middle part of this chain of inequalities, we observe that this is equivalent to $A \leq C$.

(b) *Let $\lambda \in \mathbb{R}$. Denoting $(\lambda)_+ := \max\{\lambda, 0\} \geq 0$ and $(\lambda)_- := \max\{-\lambda, 0\} = (-\lambda)_+ \geq 0$, we observe that $\lambda = (\lambda)_+ - (\lambda)_-$ and $|\lambda| = (\lambda)_+ + (\lambda)_-$. With this we obtain from functional calculus that*

$$\pm(A-B) \;=\; \pm(A-B)_+ \mp (A-B)_- \;\leq\; (A-B)_+ + (A-B)_- \;=\; |A-B|\,, \tag{3}$$

which implies that

$$D \;=\; \tfrac{1}{2}(A+B) + \tfrac{1}{2}|A-B| \;\geq\; \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B) \;=\; A\,, \tag{4}$$

$$D \;=\; \tfrac{1}{2}(A+B) + \tfrac{1}{2}|A-B| \;\geq\; \tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B) \;=\; B\,. \tag{5}$$

(c) *Clearly, $A + B = \mathbf{1}$ and $A - B = \sigma^{(3)}$ which gives $|A-B| = \mathbf{1}$, so $D = \mathbf{1}$. Moreover,*

$$A \;=\; \tfrac{1}{2}\mathbf{1} + \tfrac{1}{2}\vec{e}_3 \cdot \vec{\sigma} \quad \text{and} \quad B \;=\; \tfrac{1}{2}\mathbf{1} - \tfrac{1}{2}\vec{e}_3 \cdot \vec{\sigma}\,. \tag{6}$$

Let $r, s \in \mathbb{R}$ and $E := (r+1)\,\mathbf{1} + \frac{1}{2}s\vec{e}_1 \cdot \vec{\sigma}\mathcal{SA}(\mathbb{C}^2)$. Then

$$E - D \;=\; r\mathbf{1} + \tfrac{1}{2}s\,\vec{e}_1 \cdot \vec{\sigma}\,, \tag{7}$$

$$E - A \;=\; (r + \tfrac{1}{2})\mathbf{1} + \tfrac{1}{2}\big(s\vec{e}_1 - \vec{e}_3\big) \cdot \vec{\sigma}\,, \tag{8}$$

$$E - B \;=\; (r + \tfrac{1}{2})\mathbf{1} + \tfrac{1}{2}\big(s\vec{e}_1 + \vec{e}_3\big) \cdot \vec{\sigma}\,. \tag{9}$$

Note that $\|s\vec{e}_1 \pm \vec{e}_3\|_{\text{eukl}} = \sqrt{1+s^2}$, so $E - A \geq 0$ and $E - B \geq 0$, provided that $\frac{1}{2}\sqrt{1+s^2} \leq r + \frac{1}{2}$ which is equivalent to

$$s^2 \;\leq\; 4r^2 + 4r\,. \tag{10}$$

Furthermore, the eigenvalues of $E - D$ are $\mu_\pm = r \pm \frac{1}{2}s$, and $\mu_- < 0$ for $s > 2r$. So, for any $r > 0$ and any

$$s \;\in\; \big(2r\,,\; \sqrt{4r^2 + 4r}\big]\,, \tag{11}$$

the matrix $E = (r+1)\,\mathbf{1} + \frac{1}{2}s\vec{e}_1 \cdot \vec{\sigma}$ has the desired properties.

Problem 6.2 (6 Points):

(a) Let $x_1, x_2, y_1, y_2 \in [-1, 1]$ be real numbers. Show that

$$|x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2| \;\leq\; 2\,. \tag{12}$$

(b) Let Ω be a probability space, $f_1, f_2, g_1, g_2 : \Omega \mapsto [-1, 1]$ random variables and μ a probability measure on Ω. Prove that the correlations of these random variables satisfy

$$\big|\mathbb{E}[f_1g_1] + \mathbb{E}[f_1g_2] + \mathbb{E}[f_2g_1] - \mathbb{E}[f_2g_2]\big| \;\leq\; 2\,, \tag{13}$$

where

$$\mathbb{E}[f] \;=\; \int_\Omega f(\omega)\, d\mu(\omega) \tag{14}$$

denotes the expectation value of a random variable $f : \Omega \to \mathbb{R}$ w.r.t. μ.

Solution.

(a) *For $-1 \leq x_1, x_2, y_1, y_2 \leq 1$, we have*

$$\begin{aligned}
|x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2| \;&\leq\; |x_1| \cdot |y_1 + y_2| + |x_2| \cdot |y_1 - y_2| \;\leq\; |y_1 + y_2| + |y_1 - y_2| \\
&= \mathbf{1}[y_1 + y_2 \geq 0] \cdot \mathbf{1}[y_1 - y_2 \geq 0] \cdot 2y_1 + \mathbf{1}[y_1 + y_2 \geq 0] \cdot \mathbf{1}[y_1 - y_2 < 0] \cdot 2y_2 \\
&\quad + \mathbf{1}[y_1 + y_2 < 0] \cdot \mathbf{1}[y_1 - y_2 \geq 0] \cdot (-2y_2) + \mathbf{1}[y_1 + y_2 < 0] \cdot \mathbf{1}[y_1 - y_2 < 0] \cdot (-2y_1) \\
&\leq 2\max\big\{|y_1|\,,\; |y_2|\big\} \;\leq\; 2\,.
\end{aligned} \tag{15}$$

Alternatively, one observes that $f : \mathbb{R}^4 \to \mathbb{R}_0^+$ given by $f(x_1, x_2, y_1, y_2) := |x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2 y_2|$ is the composition of the linear map $(x_1, x_2, y_1, y_2) \mapsto x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2 y_2$ and the convex map $\lambda \mapsto |\lambda|$ and, hence, convex itself. Therefore, the maximum of f restricted to the compact convex subset $[-1,1]^4 \subseteq \mathbb{R}^4$ is attained on the extreme points of $[-1,1]^4$, i.e., on $\{-1,1\}^4$. But then $x_1^2 = x_2^2 = y_1^2 = y_2^2 = 1$ and $x_{12} = x_1 x_2, y_{12} = y_1 y_2 \in \{-1,1\}$. Inspecting the value of f on these four possibilities yields

$$|x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2\, y_2| \; = |1 + y_{12} + x_{12} - x_{12} y_{12}| \; \in \; \{0, 2\}\,, \tag{16}$$

(b) By (15)*, we have that*

$$\begin{aligned} \big|\mathbb{E}[f_1\, g_1] + \mathbb{E}[f_1\, g_2] + \mathbb{E}[f_2\, g_1] - \mathbb{E}[f_2\, g_2]\big| \;&=\; \big|\mathbb{E}[f_1\, g_1 + f_1\, g_2 + f_2\, g_1 + f_2\, g_2]\big| \\ \le\; \mathbb{E}\big(|f_1\, g_1 + f_1\, g_2 + f_2\, g_1 + f_2\, g_2|\big) \;&\le\; \mathbb{E}[2\cdot\mathbf{1}] \;=\; 2\,. \end{aligned} \tag{17}$$

Problem 6.3 (12 Points): Assume that we are given a system of two qubits, $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ which is prepared in the pure state $\rho = |\psi\rangle\langle\psi| \in \mathcal{DM}(\mathcal{H})$, with

$$\psi = \frac{1}{\sqrt{2}}\big(\uparrow\otimes\downarrow - \downarrow\otimes\uparrow\big) \;\in\; \mathcal{H}\,, \tag{18}$$

where

$$\uparrow := \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \downarrow := \begin{pmatrix}0\\1\end{pmatrix}. \tag{19}$$

For $\vec{a} \in \mathbb{R}^3$, we define

$$\vec{\sigma}\cdot\vec{a} \;:=\; a_1\sigma^{(1)} + a_2\sigma^{(2)} + a_3\sigma^{(3)}\,, \tag{20}$$

where $\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)} \in \mathcal{SA}(\mathbb{C}^2)$ denote the Pauli matrices.

(a) Compute the expectation values w.r.t. ρ of $\vec{\sigma}\cdot\vec{a}\otimes\mathbf{1}$ and $\mathbf{1}\otimes\vec{\sigma}\cdot\vec{a}$.

(b) Let $\vec{a}, \vec{b} \in \mathbb{R}^3$ be two normalized vectors, $\|\vec{a}\|_{\mathrm{eukl}} = \|\vec{b}\|_{\mathrm{eukl}} = 1$. Show that

$$\big\langle \vec{\sigma}\cdot\vec{a}\otimes\vec{\sigma}\cdot\vec{b}\big\rangle_\rho \;=\; -\vec{a}\cdot\vec{b}\,. \tag{21}$$

(c) Find four unit vectors $\vec{a}_1, \vec{a}_2, \vec{b}_1, \vec{b}_2 \in \mathbb{R}^3$ such that the expectation values of $\vec{\sigma}\cdot\vec{a}_i\otimes\vec{\sigma}\cdot\vec{b}_j$, for $i, j \in \{1,2\}$, violate inequality (13).

Solution.
We denote $\mathcal{H}_{12} := \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ with $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2$, $\rho_{12} := \rho = |\psi\rangle\langle\psi| \in \mathcal{DM}(\mathcal{H}_{12})$, $\rho_1 := \mathrm{Tr}_2\rho_{12} \in \mathcal{DM}(\mathcal{H}_1)$, and $\rho_2 := \mathrm{Tr}_1\rho_{12} \in \mathcal{DM}(\mathcal{H}_2)$. Note that

$$\psi \;=\; \frac{1}{\sqrt{2}}\big(\uparrow\otimes\downarrow - \downarrow\otimes\uparrow\big) = \frac{1}{\sqrt{2}}\sum_\tau (-1)^\tau\, \tau\otimes\overline{\tau}\,, \tag{22}$$

where the sum ranges over $\tau \in \{\uparrow,\downarrow\}$, $(-1)^\uparrow = 1$, $(-1)^\downarrow = -1$, $\overline{\uparrow} :=\downarrow$, and $\overline{\downarrow} :=\uparrow$.

(a) We compute

$$\begin{aligned} \big\langle\vec{\sigma}\cdot\vec{a}\big\rangle_{\rho_1} \;&=\; \big\langle\vec{\sigma}\cdot\vec{a}\otimes\mathbf{1}\big\rangle_{\rho_{12}} \;=\; \big\langle\psi\big|\,(\vec{\sigma}\cdot\vec{a}\otimes\mathbf{1})\psi\big\rangle_{\mathcal{H}_{12}} \\ &= \frac{1}{2}\sum_{\tau,\kappa}(-1)^\tau\,(-1)^\kappa\,\big\langle\tau\otimes\overline{\tau}\,\big|\,(\vec{\sigma}\cdot\vec{a}\otimes\mathbf{1})\kappa\otimes\overline{\kappa}\big\rangle_{\mathcal{H}_{12}} \\ &= \frac{1}{2}\sum_{\tau,\kappa}(-1)^\tau\,(-1)^\kappa\,\langle\tau\,|\,(\vec{\sigma}\cdot\vec{a})\,\kappa\rangle_{\mathcal{H}_1}\,\delta_{\overline{\tau},\overline{\kappa}} \;=\; \frac{1}{2}\sum_\tau\langle\tau\,|\,(\vec{\sigma}\cdot\vec{a})\,\tau\rangle_{\mathcal{H}_1} \\ &= \frac{1}{2}\mathrm{Tr}_{\mathcal{H}_1}\big[\vec{\sigma}\cdot\vec{a}\big] \;=\; 0 \end{aligned} \tag{23}$$

and

$$
\begin{aligned}
\left\langle \vec\sigma \cdot \vec b \right\rangle_{\rho_2} &= \left\langle \mathbf{1} \otimes \vec\sigma \cdot \vec b \right\rangle_{\rho_{12}} \;=\; \left\langle \psi \middle| \, (\mathbf{1} \otimes \vec\sigma \cdot \vec b)\psi \right\rangle_{\mathcal{H}_{12}} \\
&= \frac{1}{2} \sum_{\tau,\kappa} (-1)^{\tau} \, (-1)^{\kappa} \left\langle \tau \otimes \overline{\tau} \middle| \, (\mathbf{1} \otimes \vec\sigma \cdot \vec b)\kappa \otimes \overline{\kappa} \right\rangle_{\mathcal{H}_{12}} \\
&= \frac{1}{2} \sum_{\tau,\kappa} (-1)^{\tau} \, (-1)^{\kappa} \, \delta_{\tau,\kappa} \, \langle \overline{\tau} \, | \, (\vec\sigma \cdot \vec b) \, \overline{\kappa} \rangle_{\mathcal{H}_2} \;=\; \frac{1}{2} \sum_{\tau} \langle \overline{\tau} \, | \, (\vec\sigma \cdot \vec b) \, \overline{\tau} \rangle_{\mathcal{H}_2} \\
&= \frac{1}{2} \mathrm{Tr}_{\mathcal{H}_2} \big[\vec\sigma \cdot \vec b \big] \;=\; 0 \, . \qquad (24)
\end{aligned}
$$

(b) *Similarly, we compute*

$$
\begin{aligned}
&\left\langle \vec\sigma \cdot \vec a \otimes \vec\sigma \cdot \vec b \right\rangle_{\rho_{12}} \;=\; \left\langle \psi \middle| \, (\vec\sigma \cdot \vec a \otimes \vec\sigma \cdot \vec b)\psi \right\rangle_{\mathcal{H}_{12}} \\
&= \frac{1}{2} \sum_{\tau,\kappa} (-1)^{\tau} \, (-1)^{\kappa} \left\langle \tau \otimes \overline{\tau} \middle| \, (\vec\sigma \cdot \vec a \otimes \vec\sigma \cdot \vec b)\kappa \otimes \overline{\kappa} \right\rangle_{\mathcal{H}_{12}} \\
&= \frac{1}{2} \sum_{\tau,\kappa} (-1)^{\tau} \, (-1)^{\kappa} \, \langle \tau \, | \, (\vec\sigma \cdot \vec a) \, \kappa \rangle_{\mathcal{H}_1} \, \langle \overline{\tau} \, | \, (\vec\sigma \cdot \vec b) \, \overline{\kappa} \rangle_{\mathcal{H}_2} \\
&= \frac{1}{2} \sum_{\tau,\kappa} (-1)^{\tau} \, (-1)^{\kappa} \, \langle \tau \, | \, (\vec\sigma \cdot \vec a) \, \kappa \rangle_{\mathcal{H}_1} \, \langle \overline{\tau} \, | \, (\vec\sigma \cdot \vec b) \, \overline{\kappa} \rangle_{\mathcal{H}_2} \\
&= \frac{1}{2} \Big(\langle \uparrow | (\vec\sigma \cdot \vec a) \uparrow \rangle_{\mathcal{H}_1} \langle \downarrow | (\vec\sigma \cdot \vec b) \downarrow \rangle_{\mathcal{H}_2} \;+\; \langle \downarrow | (\vec\sigma \cdot \vec a) \downarrow \rangle_{\mathcal{H}_1} \langle \uparrow | (\vec\sigma \cdot \vec b) \uparrow \rangle_{\mathcal{H}_2} \\
&\quad - \langle \uparrow | (\vec\sigma \cdot \vec a) \downarrow \rangle_{\mathcal{H}_1} \langle \downarrow | (\vec\sigma \cdot \vec b) \uparrow \rangle_{\mathcal{H}_2} \;-\; \langle \downarrow | (\vec\sigma \cdot \vec a) \uparrow \rangle_{\mathcal{H}_1} \langle \uparrow | (\vec\sigma \cdot \vec b) \downarrow \rangle_{\mathcal{H}_2} \Big) \\
&= \frac{1}{2} \Big(a_3 \, (-b_3) \;+\; (-a_3) \, b_3 \;-\; (a_1 - i a_2)(b_1 + i b_2) \;-\; (a_1 + i a_2)(b_1 - i b_2) \Big) \\
&= \; - \vec a \cdot \vec b \, . \qquad (25)
\end{aligned}
$$

(c) *Define four unit vectors* $\vec a_1, \vec a_2, \vec b_1, \vec b_2 \in \mathbb{R}^3$ *as*

$$
\vec a_1 \;:=\; \vec e_1 \, , \quad \vec a_2 \;:=\; \vec e_2 \, , \quad \vec b_1 \;:=\; \tfrac{1}{\sqrt 2}\big(- \vec e_1 - \vec e_2 \big) \, , \quad \vec b_2 \;:=\; \tfrac{1}{\sqrt 2}\big(- \vec e_1 + \vec e_2 \big) \, . \qquad (26)
$$

Then $-\mathbf{1} \leq \vec\sigma \cdot \vec a_1, \vec\sigma \cdot \vec a_2, \vec\sigma \cdot \vec b_1, \vec\sigma \cdot \vec b_2 \leq \mathbf{1}$*, since* $(\vec\sigma \cdot \vec v)^2 = \vec v^2 \mathbf{1} \leq \mathbf{1}$*. Note that*

$$
\vec a_1 \cdot \vec b_1 \;=\; -\frac{1}{\sqrt 2} \, , \quad \vec a_1 \cdot \vec b_2 \;=\; -\frac{1}{\sqrt 2} \, , \quad \vec a_2 \cdot \vec b_1 \;=\; -\frac{1}{\sqrt 2} \, , \quad \vec a_2 \cdot \vec b_2 \;=\; \frac{1}{\sqrt 2} \, . \qquad (27)
$$

Using (25) *and* (26)*, we observe that*

$$
\begin{aligned}
&\Big| \left\langle \vec\sigma \cdot \vec a_1 \otimes \vec\sigma \cdot \vec b_1 \right\rangle_{\rho_{12}} + \left\langle \vec\sigma \cdot \vec a_1 \otimes \vec\sigma \cdot \vec b_2 \right\rangle_{\rho_{12}} + \left\langle \vec\sigma \cdot \vec a_2 \otimes \vec\sigma \cdot \vec b_1 \right\rangle_{\rho_{12}} - \left\langle \vec\sigma \cdot \vec a_2 \otimes \vec\sigma \cdot \vec b_2 \right\rangle_{\rho_{12}} \Big| \\
&\qquad = \frac{4}{\sqrt 2} \;=\; 2\sqrt 2 \, . \qquad (28)
\end{aligned}
$$

This violates Inequality (13)*, which implies that, in general, no (classical) probability space* Ω *with probabilty measure* μ *exists such that quantum mechanical expectation values* $\langle \cdot \rangle_{\rho}$ *are given by (classical) expectation values* $\mathbb{E}[\cdot] = \int_{\Omega} (\cdot) \, d\mu$*.*