Homework Problem Set 5 for the Lecture
Introduction to Quantum Information Theory

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- problem sheet uploaded on 20-Jun-2025.
- admissible format of homework is a scan of a handwritten document converted to PDF,
- submission of homework by e-mail to v.bach@tu-bs.de until 01-Jul-2025,
- discussion of the solution in the tutorial on 04-Jul-2025.

Problem 5.1 (12 Points): Let $\big(\mathcal{H}_1, \langle\cdot|\cdot\rangle_1\big)$ and $\big(\mathcal{H}_2, \langle\cdot|\cdot\rangle_2\big)$ be two complex Hilbert spaces of dimensions $\dim[\mathcal{H}_1] = d_1, \dim[\mathcal{H}_2] = d_2 \in \mathbb{N}$ and with ONB $\{f_k | k \in \mathbb{Z}_1^{d_1}\} \subseteq \mathcal{H}_1$ and $\{g_\ell | \ell \in \mathbb{Z}_1^{d_2}\} \subseteq \mathcal{H}_2$, respectively. Let furthermore $\mathcal{H}_{12} := \mathcal{H}_1 \otimes \mathcal{H}_2$ denote the corresponding tensor product Hilbert space.

(a) Show that $\{f_k \otimes g_\ell | k \in \mathbb{Z}_1^d,\ \ell \in \mathbb{Z}_1^{d'}\} \subseteq \mathcal{H}_{12}$ is an ONB and compute $\dim[\mathcal{H}_{12}]$.

(b) Given a linear operator $A \in \mathcal{B}(\mathcal{H}_{12})$ we define its *partial trace (with respect to $\mathcal{H}_2$)* $\mathrm{Tr}_2[A] \in \mathcal{B}(\mathcal{H}_1)$ by

$$\forall\, f, f' \in \mathcal{H}_1 : \quad \big\langle f \big| \mathrm{Tr}_2[A]\, f' \big\rangle_1 \ := \ \sum_{\ell=1}^{d_2} \big\langle f \otimes g_\ell \big| A(f' \otimes g_\ell) \big\rangle_{12} . \tag{1}$$

Show that (1) defines indeed a linear operator on $\mathcal{H}_1$ and that (1) is independent of choice of the ONB $\{g_\ell | \ell \in \mathbb{Z}_1^{d_2}\} \subseteq \mathcal{H}_2$.

(c) Let $\rho_{12} \in \mathcal{DM}(\mathcal{H}_{12})$ be a density matrix on $\mathcal{H}_{12}$. Show that $\rho_1 := \mathrm{Tr}_2[\rho_{12}] \in \mathcal{DM}(\mathcal{H}_1)$, i.e., that ρ_1 is a density matrix on $\mathcal{H}_1$.

(d) Let $\rho_1 \in \mathcal{DM}(\mathcal{H}_1)$ and $\rho_2 \in \mathcal{DM}(\mathcal{H}_2)$ be density matrices on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Define $\rho_1 \otimes \rho_2 \in \mathcal{B}(\mathcal{H}_{12})$ by $[\rho_1 \otimes \rho_2](f \otimes g) := (\rho_1 f) \otimes (\rho_2 g)$, for all $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$, and its linear continuation. Show that $\rho_1 \otimes \rho_2 \in \mathcal{DM}(\mathcal{H}_{12})$, i.e., that $\rho_1 \otimes \rho_2$ is a density matrix on $\mathcal{H}_{12}$.

Problem 5.2 (6 Points): Let $\big(\mathcal{H}, \langle\cdot|\cdot\rangle\big)$ be a complex Hilbert space, $U \in \mathcal{U}(\mathcal{H})$ a unitary operator and $\rho \in \mathcal{DM}(\mathcal{H})$ a density matrix on $\mathcal{H}$.

(a) Prove that $U\rho U^* \in \mathcal{DM}(\mathcal{H})$ is a density matrix, too.

(b) Prove that $U\rho U^* \in \mathcal{DM}(\mathcal{H})$ is pure if, and only if, $\rho \in \mathcal{DM}(\mathcal{H})$ is pure.

Problem 5.3 (6 Points): Denote by $\uparrow := (1,0)^T, \downarrow := (0,1)^T \in \mathbb{C}^2$ the canonical basis vectors and let

$$\Psi_{12} \ = \ \frac{1}{\sqrt{2}} \Big(\uparrow \otimes \uparrow + \downarrow \otimes \downarrow \Big) \ \in \ \mathcal{H}_{12} \ := \ \mathcal{H}_1 \otimes \mathcal{H}_2 , \tag{2}$$

where $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2$. Define $\rho_{12} := |\Psi_{12}\rangle\langle\Psi_{12}|$ and compute $\rho_1 := \mathrm{Tr}_2[\rho_{12}]$ and its Bloch representation.