Homework Problem Set 4 for the Lecture
Introduction to Quantum Information Theory

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- problem sheet uploaded on 23-May-2025.
- admissible format of homework is a scan of a handwritten document converted to PDF,
- submission of homework by e-mail to v.bach@tu-bs.de until 06-Jun-2025,
- discussion of the solution in the tutorial on 20-Jun-2025.

In this exercise we study the notion of *convexity.* Let X be a vector space over $\mathbb{K}$. A subset $M \subseteq X$ is called **convex**, if for all $p \in [0,1]$ and all $x, y \in M$ we have that

$$p\,x \;+\; (1-p)\,y \;\in\; M\,. \tag{1}$$

Let $M \subseteq X$ be a convex set. A function $F : M \to \mathbb{R}$ is called **convex** on M, if for all $p \in [0,1]$ and all $x, y \in M$ we have that

$$F\big[p\,x \;+\; (1-p)\,y\big] \;\leq\; p\,F[x] \;+\; (1-p)\,F[y]\,. \tag{2}$$

F is called **concave**, if $-F$ is convex.

Problem 4.1 (6 Points): Let X be a $\mathbb{K}$-vector space, $M \subseteq X$ be a convex subset and $F : M \to \mathbb{R}$ be a convex function on M. Prove *Jensen's inequality* for the following special case: Let $N \in \mathbb{N}$ and $p_1, p_2, ..., p_N \geq 0$ a probability distribution on $\mathbb{Z}_1^N$, i.e., $\sum_{n=1}^N p_n = 1$. and $x_1, x_2, \ldots, x_N \in M$ be an arbitrary collection of points in M. Then

$$F\Big[\sum_{n=1}^N p_n\,x_n\Big] \;\leq\; \sum_{n=1}^N p_n\,F[x_n]\,. \tag{3}$$

Hint: Induction in N.

Solution.
We may assume w.l.o.g. that $p_1, \ldots, p_N > 0$ are strictly positive. For $N = 1$ there is nothing to prove. For $N = 2$ we have that $p_2 = 1 - p_1$ and the assertion follows from (2) with $p := p_1$,

$$\begin{aligned} F[p_1 x_1 + p_2 x_2] \;&=\; F[p_1 x_1 + (1-p_1) x_2] \\ &\leq p_1\,F[x_1] \;+\; (1-p_1) F[x_2] \;=\; p_1\,F[x_1] \;+\; p_2\,F[x_2]\,. \end{aligned} \tag{4}$$

Now assume (3) to hold true for $N - 1 \geq 2$ and define

$$p \;:=\; \sum_{n=1}^{N-1} p_n \quad \Rightarrow \quad 1 - p = p_N\,. \tag{5}$$

Eq. (2) implies that

$$\begin{aligned} F\Big[\sum_{n=1}^N p_n\,x_n\Big] \;&=\; F\Big[p\Big(\sum_{n=1}^{N-1} \frac{p_n}{p}\,x_n\Big) \;+\; (1-p) x_N\Big] \;\leq\; p\,F\Big[\sum_{n=1}^{N-1} \frac{p_n}{p}\,x_n\Big] \;+\; (1-p) F[x_N] \\ &= p\,F\Big[\sum_{n=1}^{N-1} \frac{p_n}{p}\,x_n\Big] \;+\; p_N\,F[x_N]\,. \end{aligned} \tag{6}$$

Moreover, $\{\frac{p_1}{p}, \ldots, \frac{p_1}{p}\} \subseteq (0,1)$ *is a probability distribution of* $N-1$ *weights, and by induction we obtain*

$$p\,F\left[\sum_{n=1}^{N-1} \frac{p_n}{p}\, x_n\right] \;+\; p_N\, F[x_N] \;\leq p \sum_{n=1}^{N-1} \frac{p_n}{p}\, F[x_n] \;+\; p_N\, F[x_N] \tag{7}$$

$$= \left(\sum_{n=1}^{N-1} p_n\, F[x_n]\right) \;+\; p_N\, F[x_N] \;=\; \sum_{n=1}^{N} p_n\, F[x_n]\,.$$

Problem 4.2 (12 Points): Let $d \in \mathbb{N}$ and $\mathcal{H} = \mathbb{C}^d$ be the complex d-dimensional Hilbert space with unitary scalar product. Furthermore let $F : \mathbb{R} \to \mathbb{R}$ be a convex function.

(i) Show that the subset $\mathcal{SA}(\mathcal{H}) := \{A \in \mathcal{B}(\mathcal{H}) | A = A^*\} \subseteq \mathcal{B}(\mathcal{H})$ of self-adjoint operators on $\mathcal{H}$ is a convex subset of $\mathcal{B}(\mathcal{H})$.

(ii) Let $A \in \mathcal{SA}(\mathcal{H})$ be a self-adjoint operator on $\mathcal{H}$ and $x \in \mathcal{H}$ a normalized vector. Show that

$$F\big[\langle x | Ax \rangle\big] \;\leq\; \langle x \,|\, F[A]\, x \rangle\,. \tag{8}$$

(iii) Show that the map $\mathcal{SA}(\mathcal{H}) \to \mathbb{R}$, $A \mapsto \mathrm{Tr}[F(A)]$, is convex.

Solution.

(i) Let $A, B \in \mathcal{SA}(\mathcal{H})$ *and* $p \in [0,1]$*. Then* $[pA + (1-p)B]^* = \overline{p}A^* + \overline{(1-p)}B^* = pA + (1-p)B$ *and hence* $pA + (1-p)B \in \mathcal{SA}(\mathcal{H})$*, so* $\mathcal{SA}(\mathcal{H})$ *is convex, indeed.*

(ii) Since $A = A^*$ *is self-adjoint, there is an ONB* $\{\varphi_j\}_{j=1}^d \subseteq \mathcal{H}$ *of eigenvectors with corresponding eigenvalues* $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ *such that*

$$A \;=\; \sum_{j=1}^{d} \lambda_j\, |\varphi_j\rangle\langle\varphi_j|\,. \tag{9}$$

It follows that

$$\langle x | Ax \rangle \;=\; \sum_{j=1}^{d} \lambda_j\, \langle x | \varphi_j \rangle\, \langle \varphi_j | x \rangle \;=\; \sum_{j=1}^{d} \lambda_j\, |\langle x | \varphi_j \rangle|^2\,. \tag{10}$$

Moreover, since $x \in \mathcal{H}$ *is normalized,*

$$\sum_{j=1}^{d} |\langle x | \varphi_j \rangle|^2 \;=\; \sum_{j=1}^{d} \langle x | \varphi_j \rangle\, \langle \varphi_j | x \rangle \;=\; \langle x | x \rangle \;=\; 1\,, \tag{11}$$

and $p_1, \ldots, p_d \in [0,1]$ *given by* $p_j := |\langle x | \varphi_j \rangle|^2$ *define a probability distribution. Applying Jensen's inequality* (3) *with these weights and random variables* $\lambda_1, \ldots, \lambda_d$*, we obtain*

$$F\big[\langle x | Ax \rangle\big] \;=\; F\left[\sum_{j=1}^{d} |\langle x | \varphi_j \rangle|^2\, \lambda_j\right] \;\leq\; \sum_{j=1}^{d} |\langle x | \varphi_j \rangle|^2\, F[\lambda_j]$$

$$= \left\langle x \,\middle|\, \Big(\sum_{j=1}^{d} F[\lambda_j]\, |\varphi_j\rangle\langle\varphi_j|\Big) x \right\rangle \;=\; \langle x \,|\, F[A]\, x \rangle\,, \tag{12}$$

by the spectral theorem (functional calculus).

(iii) Let $A, B \in \mathcal{SA}(\mathcal{H})$ *and* $p \in (0,1)$*. Since* $pA + (1-p)B$ *is self-adjoint, there exist an ONB* $\{x_j\}_{j=1}^d \subseteq \mathcal{H}$ *of eigenvectors of* $pA + (1-p)B$ *with corresponding eigenvalues* $\mu_1, \ldots, \mu_d \in \mathbb{R}$*.*

We use this basis to calculate the trace of $F[pA+(1-p)B]$, which is the sum of its eigenvalues $F[\mu_j]$,

$$
\begin{aligned}
\mathrm{Tr}\big(F[pA+(1-p)B]\big) \;&=\; \sum_{j=1}^{d} F[\mu_j] \;=\; \sum_{j=1}^{d} F\big[\langle x_j|[pA+(1-p)B]\,x_j\rangle\big] \\
&=\; \sum_{j=1}^{d} F\big[p\,\langle x_j|Ax_j\rangle \;+\; (1-p)\langle x_j|Bx_j\rangle\big] \qquad (13)\\
&\leq\; p\sum_{j=1}^{d} F\big[\langle x_j|Ax_j\rangle\big] \;+\; (1-p)\sum_{j=1}^{d} F\big[\langle x_j|Bx_j\rangle\big]\,,
\end{aligned}
$$

where the inequality is a consequence of the convexity of F. On the other hand, for $j \in \mathbb{Z}_1^d$, Eq. (8) implies that

$$
F\big[\langle x_j|Ax_j\rangle\big] \;\leq\; \big\langle x_j\big|F[A]\,x_j\big\rangle \quad \text{and} \quad F\big[\langle x_j|Bx_j\rangle\big] \;\leq\; \big\langle x_j\big|F[B]\,x_j\big\rangle\,. \qquad (14)
$$

Summing these inequalities over $j \in \mathbb{Z}_1^d$, we arrive at

$$
\begin{aligned}
\mathrm{Tr}\big(F[pA+(1-p)B]\big) \;&\leq\; p\sum_{j=1}^{d} F\big[\langle x_j|Ax_j\rangle\big] \;+\; (1-p)\sum_{j=1}^{d} F\big[\langle x_j|Bx_j\rangle\big] \\
&\leq\; p\sum_{j=1}^{d} \big\langle x_j\big|F[A]\,x_j\big\rangle \;+\; (1-p)\sum_{j=1}^{d} \big\langle x_j\big|F[B]\,x_j\big\rangle \\
&=\; p\,\mathrm{Tr}\big(F[A]\big) \;+\; (1-p)\,\mathrm{Tr}\big(F[B]\big)\,, \qquad (15)
\end{aligned}
$$

which is the asserted convexity of $A \mapsto \mathrm{Tr}\big(F[A]\big)$.

Problem 4.3 (6 Points): Let $d \in \mathbb{N}$ and $\mathcal{H} = \mathbb{C}^d$ be the complex d-dimensional Hilbert space with unitary scalar product and $A, B \in \mathcal{SA}(\mathcal{H})$ be two positive operators. Show that

$$
\mathrm{Tr}(AB) \;\geq\; 0 \quad \text{and that} \quad \Big\{\mathrm{Tr}(AB) = 0 \;\Rightarrow\; AB = 0\Big\}\,. \qquad (16)
$$

Solution.
Since $A = A^* \geq 0$ is self-adjoint and positive, there is an ONB $\{\varphi_j\}_{j=1}^d \subseteq \mathcal{H}$ of eigenvectors with corresponding eigenvalues $\lambda_1, \ldots, \lambda_d \geq 0$ such that

$$
A \;=\; \sum_{j=1}^{d} \lambda_j\,|\varphi_j\rangle\langle\varphi_j|\,. \qquad (17)
$$

Using this basis to compute the trace of AB, we obtain

$$
\mathrm{Tr}(AB) \;=\; \sum_{j=1}^{d} \lambda_j\,\langle\varphi_j|B\varphi_j\rangle \;\geq\; 0\,, \qquad (18)
$$

since $\langle\varphi_j|B\varphi_j\rangle \geq 0$, as $B = B^* \geq 0$ is positive, too.

Since $\lambda_j\,\langle\varphi_j|B\varphi_j\rangle \geq 0$, for every $j \in \mathbb{Z}_1^d$, we observe that $\mathrm{Tr}(AB) = 0$ implies that

$$
\forall\, j \in \mathbb{Z}_1^d : \quad \lambda_j\,\big\|\sqrt{B}\varphi_j\big\|^2 \;=\; \lambda_j\,\langle\varphi_j|B\varphi_j\rangle \;=\; 0\,, \qquad (19)
$$

where $\sqrt{B} \geq 0$ denotes the square-root of B (defined by functional calculus). Hence, for $k, \ell \in \mathbb{Z}_1^d$, we have that

$$
|\langle\varphi_k|AB\,\varphi_\ell\rangle|^2 \;=\; \lambda_k^2\,|\langle\varphi_k|B\,\varphi_\ell\rangle|^2 \;=\; \lambda_k^2\,|\langle\sqrt{B}\varphi_k|\sqrt{B}\varphi_\ell\rangle|^2 \;\leq\; \lambda_k^2\,\big\|\sqrt{B}\varphi_k\big\|\,\big\|\sqrt{B}\varphi_\ell\big\| \;=\; 0\,, \qquad (20)
$$

by (19). Thus, all matrix elements $\langle\varphi_k|AB\,\varphi_\ell\rangle$ of AB w.r.t. the ONB $\{\varphi_j\}_{j=1}^d \subseteq \mathcal{H}$ vanish. Since any two vectors $\psi, \phi \in \mathcal{H}$ are linear combinations of these basis vectors, it follows that

$$
\forall\, \psi, \phi \in \mathcal{H} : \quad \langle\psi|\,AB\,\phi\rangle \;=\; 0\,, \qquad (21)
$$

which is equivalent to $AB = 0$.