Homework Problem Set 2 for the Lecture
Introduction to Quantum Information Theory

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- problem sheet uploaded on 17-Apr-2025.
- admissible format of homework is a scan of a handwritten document converted to PDF,
- submission of homework by e-mail to v.bach@tu-bs.de until 01-May-2025,
- discussion of the solution in the tutorial on 08-May-2025.

Throughout this exercise, $N \in \mathbb{N}$ is a positive integer, and $\mathcal{H} = \mathbb{C}^N$ is the N-dimensional complex Hilbert space with the usual unitary scalar product

$$\forall \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix} \in \mathbb{C}^N : \quad \left\langle \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} \middle| \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix} \right\rangle := \sum_{j=1}^N \overline{\alpha_j}\, \beta_j \, . \tag{1}$$

In this case, the space $\mathcal{B}(\mathcal{H}) = \{A : \mathcal{H} \to \mathcal{H} | A \text{ is linear}\}$ of (bounded) linear operators (= linear maps) on $\mathcal{H}$ can be identified with the space of complex $N \times N$ matrices by identifying a linear map $A \in \mathcal{B}(\mathcal{H})$ with its matrix representation $(A_{i,j})_{i,j=1}^N \in \mathbb{C}^{N \times N}$ w.r.t. the canonical ONB $\mathcal{E} = \{e_1, e_2, \ldots, e_N\}$, where the canonical basis vectors in $\mathcal{H}$ are given by

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \; \ldots, \; e_N = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{2}$$

While we usually simply identify $\mathcal{B}(\mathcal{H})$ with $\mathbb{C}^{N \times N}$, we deliberately distinguish this difference in this problem set.

Problem 2.1 (6 Points):

(a) Show that $\mathcal{E} = \{e_j\}_{j=1}^N$ is an orthonormal basis (ONB).

(b) Let $\psi = \sum_{j=1}^N \psi_j \, e_j \in \mathcal{H}$. Show that $\psi_j = \langle e_j | \psi \rangle$, for all $j \in \mathbb{Z}_1^N := \{1, 2, \ldots, N\}$.

(c) Let $A \in \mathcal{B}(\mathcal{H})$ be a linear operator on $\mathcal{H}$ and $(A_{i,j})_{i,j=1}^N \in \mathbb{C}^{N \times N}$ its matrix representation w.r.t. the canonical ONB $\mathcal{E}$. Show that $A_{i,j} = \langle e_i | A e_j \rangle$, for all $i, j \in \mathbb{Z}_1^N$.

(d) Let $A \in \mathcal{B}(\mathcal{H})$ be a linear operator on $\mathcal{H}$, $(A_{i,j})_{i,j=1}^N \in \mathbb{C}^{N \times N}$ its matrix representation w.r.t. the canonical ONB $\mathcal{E}$, and $A^* \in \mathcal{B}(\mathcal{H})$ its adoint defined by the matrix representation $\big((A^*)_{i,j}\big)_{i,j=1}^N \in \mathbb{C}^{N \times N}$ w.r.t. the canonical ONB $\mathcal{E}$, where $(A^*)_{i,j} := \overline{A_{j,i}}$. Show that $\langle \varphi | A \psi \rangle = \langle A^* \varphi | \psi \rangle$ holds true, for all $\varphi, \psi \in \mathcal{H}$.

Solution.

(a) Since obviously $\langle e_i | e_j \rangle = \delta_{i,j}$, $\mathcal{E} \subseteq \mathbb{C}^N$ is orthonormal. Being an orthonormal set, $\mathcal{E}$ is linearly independent, for if $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$ are such that $\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_N e_N = 0$, then, for any $j \in \mathbb{Z}_1^N$,

$$0 \;=\; \langle e_j | 0 \rangle \;=\; \sum_{n=1}^{N} \alpha_n \, \langle e_j | e_n \rangle \;=\; \sum_{n=1}^{N} \alpha_n \, \delta_{j,n} \;=\; \alpha_j \,, \tag{3}$$

which implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_N = 0$. This establishes the linear independence of $\mathcal{E}$. Moreover, if $z = (z_1, z_2, \ldots, z_N)^T \in \mathbb{C}^N$ then

$$z \;=\; z_1 e_1 + z_2 e_2 + \ldots + z_N e_N \,, \tag{4}$$

and hence $\mathcal{E}$ generate $\mathbb{C}^N$. It follows that $\mathcal{E}$ is a basis.

(b) We take the scalar product of $\psi = \sum_{k=1}^{N} \psi_k \, e_k$ with e_j and obtain

$$\langle e_j | \psi \rangle \;=\; \sum_{k=1}^{N} \psi_k \, \langle e_j | e_k \rangle \;=\; \sum_{k=1}^{N} \psi_k \, \delta_{j,k} \;=\; \psi_j \,. \tag{5}$$

(c) Since $(A_{i,j})_{i,j=1}^N$ is the matrix representation of A, we have that $A e_j = \sum_{k=1}^{N} A_{k,j} \, e_k$, for all $j \in \mathbb{Z}_1^N$. We take the scalar product of this with e_i and obtain

$$\langle e_i | A e_j \rangle \;=\; \sum_{k=1}^{N} A_{k,j} \, \langle e_i | e_k \rangle \;=\; \sum_{k=1}^{N} A_{k,j} \, \delta_{i,k} \;=\; A_{i,j} \,. \tag{6}$$

(d) If $\varphi = \sum_{i=1}^{N} \varphi_i \, e_i$ and $\psi = \sum_{j=1}^{N} \psi_j \, e_j$ then by sesquilinearity

$$\langle \varphi | A \psi \rangle \;=\; \sum_{i,j=1}^{N} \overline{\varphi_i} \, \psi_j \, \langle e_i | A e_j \rangle \quad \text{and} \quad \langle A^* \varphi | \psi \rangle \;=\; \sum_{i,j=1}^{N} \overline{\varphi_i} \, \psi_j \, \langle A^* e_i | e_j \rangle \,. \tag{7}$$

Hence, it suffices to show that $\langle e_i | A e_j \rangle = \langle A^* e_i | e_j \rangle$ holds true, for all $i, j \in \mathbb{Z}_1^N$. We observe, however, that

$$\langle A^* e_i | e_j \rangle \;=\; \overline{\langle e_j | A^* e_i \rangle} \;=\; \overline{(A^*)_{j,i}} \;=\; A_{i,j} \;=\; \langle e_i | A e_j \rangle \,, \tag{8}$$

indeed.

Problem 2.2 (12 Points):

(a) Let $\mathcal{D} = \{\varphi_j\}_{j=1}^N \subseteq \mathcal{H}$ be an ONB. Define a linear operator $U \in \mathcal{B}(\mathcal{H})$ by $U e_j := \varphi_j$, for all $j \in \mathbb{Z}_1^N$. Show that the matrix elements $(U_{i,j})_{i,j=1}^N \in \mathbb{C}^{N \times N}$ of U w.r.t. the canonical ONB $\mathcal{E}$ are given by $U_{i,j} = \langle e_i | \varphi_j \rangle$.

(b) Show that U is unitary, i.e., that $UU^* = U^*U = \mathbf{1}_{\mathcal{H}}$.

(c) Let $N = 2$ and suppose that the matrix representation $(u_{i,j})_{i,j=1}^2 \in \mathbb{C}^{2 \times 2}$ of $U \in \mathcal{B}(\mathcal{H})$ w.r.t. the canonical ONB $\mathcal{E}$ is given by

$$\begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix} \;=\; \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} . \tag{9}$$

(c.1) Determine all values of $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that U is invertible.
(c.2) Compute the inverse matrix for these values of $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.
(c.3) Determine all values of $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that U is unitary.

Solution.

(a) The matrix elements $A_{i,j}$ of a linear operator A w.r.t. the canonical ONB $\mathcal{E}$ are given by $A_{i,j} = \langle e_i | A e_j \rangle$. For U this yields $U_{i,j} = \langle e_i | U e_j \rangle = \langle e_i | \varphi_j \rangle$.

(b) Using Pb 1 and (a), we obtain $(U^*)_{i,j} = \overline{U_{j,i}} = \overline{\langle e_j | \varphi_i \rangle} = \langle \varphi_i | e_j \rangle$ and hence

$$U = \sum_{i,j=1}^{N} U_{i,j} \, |e_i\rangle\langle e_j| = \sum_{i,j=1}^{N} \langle e_i | \varphi_j \rangle \, |e_i\rangle\langle e_j| \,, \tag{10}$$

$$U^* = \sum_{k,\ell=1}^{N} (U^*)_{k,\ell} \, |e_k\rangle\langle e_\ell| = \sum_{k,\ell=1}^{N} \langle \varphi_k | e_\ell \rangle \, |e_k\rangle\langle e_\ell| \,. \tag{11}$$

Hence,

$$\begin{aligned} U^*U &= \sum_{i,j,k,\ell=1}^{N} \langle \varphi_k | e_\ell \rangle \, \langle e_i | \varphi_j \rangle \, |e_k\rangle \underbrace{\langle e_\ell | e_i \rangle}_{=\delta_{\ell,i}} \langle e_j| = \sum_{i,j,k=1}^{N} \langle \varphi_k | e_i \rangle \, \langle e_i | \varphi_j \rangle \, |e_k\rangle\langle e_j| \\ &= \sum_{j,k=1}^{N} \Big\langle \varphi_k \Big| \underbrace{\Big(\sum_{i=1}^{N} |e_i\rangle\langle e_i| \Big)}_{=\mathbf{1}} \varphi_j \Big\rangle \, |e_k\rangle\langle e_j| = \sum_{j,k=1}^{N} \underbrace{\langle \varphi_k | \varphi_j \rangle}_{=\delta_{k,j}} \, |e_k\rangle\langle e_j| \\ &= \sum_{j=1}^{N} |e_j\rangle\langle e_j| = \mathbf{1} \,. \end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned} UU^* &= \sum_{i,j,k,\ell=1}^{N} \langle e_i | \varphi_j \rangle \, \langle \varphi_k | e_\ell \rangle \, |e_i\rangle \underbrace{\langle e_j | e_k \rangle}_{=\delta_{j,k}} \langle e_\ell| = \sum_{i,j,\ell=1}^{N} \langle e_i | \varphi_j \rangle \, \langle \varphi_j | e_\ell \rangle \, |e_i\rangle\langle e_\ell| \\ &= \sum_{i,\ell=1}^{N} \Big\langle e_i \Big| \underbrace{\Big(\sum_{j=1}^{N} |\varphi_j\rangle\langle \varphi_j| \Big)}_{=\mathbf{1}} e_\ell \Big\rangle \, |e_i\rangle\langle e_\ell| = \sum_{i,\ell=1}^{N} \underbrace{\langle e_i | e_\ell \rangle}_{=\delta_{i,\ell}} \, |e_j\rangle\langle e_\ell| \\ &= \sum_{i=1}^{N} |e_i\rangle\langle e_i| = \mathbf{1} \,. \end{aligned} \tag{13}$$

(c) (c.1) U is invertible if, and only if, $\det[U] = \alpha\delta - \beta\gamma \neq 0$.

(c.2) For 2×2-matrices, there is a formula for the matrix inverse, namely

$$U^{-1} = \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} . \tag{14}$$

Indeed, we can check that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha\delta - \beta\gamma & 0 \\ 0 & \alpha\delta - \beta\gamma \end{pmatrix} . \tag{15}$$

(c.3) Unitarity of U is equivalent to $U^{-1} = U^*$. First, this implies that

$$\begin{aligned} |\det[U]|^2 &= \overline{\det[U]} \, \det[U] = \overline{\det[U^T]} \, \det[U] = \det[U^*] \, \det[U] \\ &= \det[U^*U] = \det[\mathbf{1}] = 1 \,. \end{aligned} \tag{16}$$

Hence, there is a $\varphi \in [0, 2\pi)$ such that $\det[U] = e^{i\varphi}$. The unitarity further implies that

$$\begin{pmatrix} e^{-i\varphi}\delta & -e^{-i\varphi}\beta \\ -e^{-i\varphi}\gamma & e^{-i\varphi}\alpha \end{pmatrix} = U^{-1} = U^* = \begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{pmatrix} . \tag{17}$$

It follows that $\delta = \overline{e^{-i\varphi}\alpha} = e^{i\varphi}\overline{\alpha}$ and that $\gamma = \overline{e^{-i\varphi}\beta} = e^{i\varphi}\overline{\beta}$. But then

$$e^{i\varphi} = \det[U] = \alpha\delta - \beta\gamma = e^{i\varphi}\big\{ |\alpha|^2 + |\beta|^2 \big\} \,. \tag{18}$$

Hence, $\alpha = e^{i\tau}\cos(\vartheta)$ and $\beta = e^{i\sigma}\sin(\vartheta)$, for some $\vartheta, \sigma, \tau \in [0, 2\pi)$. It follows that every unitary 2×2-matrix is of the form

$$U = \begin{pmatrix} e^{i\tau}\cos(\vartheta) & e^{i\sigma}\sin(\vartheta) \\ e^{i(\varphi-\sigma)}\sin(\vartheta) & e^{i(\varphi-\tau)}\cos(\vartheta) \end{pmatrix}, \tag{19}$$

for some $\vartheta, \varphi, \tau, \sigma \in [0, 2\pi)$. Conversely, as is easily checked, every matrix of the form (19) is unitary, provided $\vartheta, \varphi, \tau, \sigma \in [0, 2\pi)$.

Problem 2.3 (6 Points): Let $A \in \mathcal{B}(\mathcal{H})$ be a linear operator on $\mathcal{H}$ and $(A_{i,j})_{i,j=1}^N \in \mathbb{C}^{N\times N}$ its matrix representation w.r.t. the canonical ONB $\mathcal{E}$. The *trace of* A is defined as $\mathrm{Tr}(A) := \sum_{j=1}^N A_{j,j}$.

(a) Show that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, for all $A, B \in \mathcal{B}(\mathcal{H})$.

(b) Let $\mathcal{D} = \{\varphi_j\}_{j=1}^N \subseteq \mathcal{H}$ be another ONB. Show that

$$\forall A \in \mathcal{B}(\mathcal{H}): \qquad \mathrm{Tr}(A) = \sum_{j=1}^N \langle \varphi_j | A\varphi_j \rangle . \tag{20}$$

Solution.

(a) Given $A, B \in \mathcal{B}(\mathcal{H})$, we first compute the matrix elements of AB,

$$\begin{aligned} (AB)_{i,j} &= \langle e_i | ABe_j \rangle = \Big\langle e_i \Big| A\Big(\sum_{k=1}^N |e_k\rangle\langle e_k|\Big)Be_j \Big\rangle = \sum_{k=1}^N \langle e_i | Ae_k \rangle \, \langle e_k | Be_j \rangle \\ &= \sum_{k=1}^N A_{i,k}\, B_{k,j} , \end{aligned} \tag{21}$$

and similarly $(BA)_{k,\ell} = \sum_{j=1}^N B_{k,j} A_{j,\ell}$. Hence,

$$\begin{aligned} \mathrm{Tr}(AB) &= \sum_{j=1}^N (AB)_{j,j} = \sum_{j,k=1}^N A_{j,k}\, B_{k,j} = \sum_{j,k=1}^N B_{k,j}\, A_{j,k} = \sum_{k=1}^N (BA)_{k,k} \\ &= \mathrm{Tr}(BA) . \end{aligned} \tag{22}$$

(b) Let $U \in \mathcal{B}(H)$ be the unitary such that $Ue_j = \varphi_j$. It follows that

$$\begin{aligned} \mathrm{Tr}[A] &= \mathrm{Tr}[(UU^*)A] = \mathrm{Tr}[U(U^*A)] = \mathrm{Tr}[(U^*A)U] = \mathrm{Tr}[U^*AU] \\ &= \sum_{j=1}^N \langle e_j | U^*AUe_j \rangle = \sum_{j=1}^N \langle Ue_j | AUe_j \rangle = \sum_{j=1}^N \langle \varphi_j | A\varphi_j \rangle , \end{aligned} \tag{23}$$

where we use $\mathrm{Tr}[XY] = \mathrm{Tr}[YX]$ in the third equation