Homework Problem Set 1 for the Lecture
Introduction to Quantum Information Theory

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- problem sheet uploaded on 08-Apr-2025.
- admissible format of homework is a scan of a handwritten document converted to PDF,
- submission of homework by e-mail to v.bach@tu-bs.de until 15-Apr-2025,
- discussion of the solution in the tutorial on 17-Apr-2025.

Problem 1.1 (6 Points): Let $\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, \vec{x}_4, \vec{y}_1, \vec{y}_2, \vec{y}_3 \in \mathbb{R}^5$ be the vectors

$$\vec{x}_1 := \begin{pmatrix} 2 \\ 0 \\ 3 \\ -1 \\ 1 \end{pmatrix}, \quad \vec{x}_2 := \begin{pmatrix} -1 \\ 0 \\ 4 \\ 4 \\ 8 \end{pmatrix}, \quad \vec{x}_3 := \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{x}_4 := \begin{pmatrix} -1 \\ 1 \\ 0 \\ 3 \\ 4 \end{pmatrix},$$

$$\vec{y}_1 := \begin{pmatrix} 3 \\ -3 \\ 5 \\ 3 \\ -1 \end{pmatrix}, \quad \vec{y}_2 := \begin{pmatrix} 1 \\ 3 \\ 1 \\ -5 \\ 3 \end{pmatrix}, \quad \vec{y}_2 := \begin{pmatrix} 0 \\ 6 \\ -1 \\ -9 \\ 5 \end{pmatrix},$$

and define the subspaces $U, V \subseteq \mathbb{R}^5$ by $U := \mathrm{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$ and $V := \mathrm{span}\{\vec{y}_1, \vec{y}_2, \vec{y}_3\}$.

(a) Determine a basis of U.

(b) Determine a basis of $U + V$.

(c) Determine a basis of $U \cap V$.

Solution.

(a) Let $\alpha_1, \ldots, \alpha_4 \in \mathbb{R}$ be real numbers such that $\alpha_1 \vec{x}_1 + \ldots + \alpha_4 \vec{x}_4 = \vec{0}$. Then $\alpha_1, \ldots, \alpha_4$ solve the system

$$\begin{array}{rrrrcl} 2\alpha_1 & -\alpha_2 & +\alpha_3 & -\alpha_4 & = & 0\,, \\ & & 2\alpha_3 & +\alpha_4 & = & 0\,, \\ 3\alpha_1 & +4\alpha_2 & -\alpha_3 & & = & 0\,, \\ -\alpha_1 & +4\alpha_2 & +\alpha_3 & +3\alpha_4 & = & 0\,, \\ \alpha_1 & +8\alpha_2 & +\alpha_3 & +4\alpha_4 & = & 0\,, \end{array} \tag{1}$$

of linear equations. From the second equation we directly obtain $\alpha_4 = -2\alpha_3$ which, inserted in the other equations, yields

$$\begin{array}{rrrcl} -\alpha_1 & +4\alpha_2 & -5\alpha_3 & = & 0\,, \\ 2\alpha_1 & -\alpha_2 & +3\alpha_3 & = & 0\,, \\ 3\alpha_1 & +4\alpha_2 & -\alpha_3 & = & 0\,, \\ \alpha_1 & +8\alpha_2 & -7\alpha_3 & = & 0\,, \end{array} \tag{2}$$

which is equivalent to

$$\begin{array}{rrrcl} -\alpha_1 & +4\alpha_2 & -5\alpha_3 & = & 0\,, \\ & 7\alpha_2 & -7\alpha_3 & = & 0\,, \\ & 16\alpha_2 & -16\alpha_3 & = & 0\,, \\ & 12\alpha_2 & -12\alpha_3 & = & 0\,. \end{array} \tag{3}$$

From this we get $\alpha_3 = \alpha_2$ and finally $\alpha_2 = -\alpha_1$. Choosing $\alpha_1 := -1$ yields $\alpha_2 = \alpha_3 = 1$ and $\alpha_4 = -2$. Thus $\vec{x}_1$ can be written as a linear combination of $\vec{x}_2, \vec{x}_3, \vec{x}_4$, namely, $\vec{x}_1 = \vec{x}_2 + \vec{x}_3 - 2\vec{x}_4$.

Moreover, our computation shows that if $\alpha_1 = 0$ then the only solution of $\alpha_2\vec{x}_2 + \alpha_3\vec{x}_3 + \alpha_4\vec{x}_4 = \vec{0}$ is $\alpha_2 = \alpha_3 = \alpha_4 = 0$. This proves that $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\} \subseteq \mathbb{R}^5$ is linearly independent. Furthermore, $U = \operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\} = \operatorname{span}\{\vec{x}_2, \vec{x}_3, \vec{x}_4\}$ and hence $\{\vec{x}_2, \vec{x}_3, \vec{x}_4\} \subseteq U$ is a basis.

(b) We first construct a basis of V. Let $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ be real numbers such that $\beta_1\vec{y}_1 + \beta_2\vec{y}_2 + \beta_3\vec{y}_3 = \vec{0}$. Then $\beta_1, \beta_2, \beta_3$ solve the system

$$\begin{array}{rrrcl} 3\beta_1 & +\beta_2 & & = & 0\,, \\ -3\beta_1 & +3\beta_2 & +6\beta_3 & = & 0\,, \\ 5\beta_1 & +\beta_2 & -\beta_3 & = & 0\,, \\ 3\beta_1 & -5\beta_2 & -9\beta_3 & = & 0\,, \\ -\beta_1 & +3\beta_2 & +5\beta_3 & = & 0\,, \end{array} \tag{4}$$

of linear equations. From the first equation we directly obtain $\beta_2 = -3\beta_1$ which, inserted in the other equations, yields

$$\begin{array}{rrcl} -12\beta_1 & +6\beta_3 & = & 0\,, \\ 2\beta_1 & -\beta_3 & = & 0\,, \\ 18\beta_1 & -9\beta_3 & = & 0\,, \\ -10\beta_1 & +5\beta_3 & = & 0\,, \end{array} \tag{5}$$

and hence $\beta_3 = 2\beta_1$. Choosing $\beta_1 := -1$, we obtain $\beta_2 = 3$, $\beta_3 = -2$, and thus $\vec{y}_1 = 3\vec{y}_2 - 2\vec{y}_3$.

Moreover, our computation shows that if $\beta_1 = 0$ then the only solution of $\beta_2\vec{y}_2 + \beta_3\vec{y}_3 = \vec{0}$ is $\beta_2 = \beta_3 = 0$. This proves that $\{\vec{y}_2, \vec{y}_3\} \subseteq \mathbb{R}^5$ is linearly independent. Furthermore, $V = \operatorname{span}\{\vec{y}_1, \vec{y}_2, \vec{y}_3\} = \operatorname{span}\{\vec{y}_2, \vec{y}_3\}$ and hence $\{\vec{y}_2, \vec{y}_3\} \subseteq U$ is a basis.

Let $\gamma_1, \ldots, \gamma_5 \in \mathbb{R}$ be real numbers such that $\gamma_1\vec{x}_2 + \gamma_2\vec{x}_3 + \gamma_3\vec{x}_4 + \gamma_4\vec{y}_2 + \gamma_5\vec{y}_3 = \vec{0}$. Then $\gamma_1, \ldots, \gamma_5$ solve the system

$$\begin{array}{rrrrrcl} -\gamma_1 & +\gamma_2 & -\gamma_3 & +\gamma_4 & & = & 0\,, \\ & 2\gamma_2 & +\gamma_3 & +3\gamma_4 & +6\gamma_5 & = & 0\,, \\ 4\gamma_1 & -\gamma_2 & & +\gamma_4 & -\gamma_5 & = & 0\,, \\ 4\gamma_1 & +\gamma_2 & +3\gamma_3 & -5\gamma_4 & -9\gamma_5 & = & 0\,, \\ 8\gamma_1 & +\gamma_2 & +4\gamma_3 & +3\gamma_4 & +5\gamma_5 & = & 0\,, \end{array} \tag{6}$$

of linear equations, which is equivalent to

$$\begin{array}{rrrrrcl} 3\gamma_1 & & -\gamma_3 & +2\gamma_4 & -\gamma_5 & = & 0\,, \\ 8\gamma_1 & & +\gamma_3 & +5\gamma_4 & +4\gamma_5 & = & 0\,, \\ 4\gamma_1 & -\gamma_2 & & +\gamma_4 & -\gamma_5 & = & 0\,, \\ 8\gamma_1 & & +3\gamma_3 & -4\gamma_4 & -10\gamma_5 & = & 0\,, \\ 3\gamma_1 & & +\gamma_3 & +\gamma_4 & +\gamma_5 & = & 0\,, \end{array} \tag{7}$$

of linear equations, which is equivalent to

$$\begin{array}{rrrrrcl} 3\gamma_1 & & -\gamma_3 & +2\gamma_4 & -\gamma_5 & = & 0\,, \\ 11\gamma_1 & & & +7\gamma_4 & +3\gamma_5 & = & 0\,, \\ 4\gamma_1 & -\gamma_2 & & +\gamma_4 & -\gamma_5 & = & 0\,, \\ 17\gamma_1 & & & +2\gamma_4 & -13\gamma_5 & = & 0\,, \\ 6\gamma_1 & & & +3\gamma_4 & & = & 0\,. \end{array} \tag{8}$$

This implies that $\gamma_4 = -2\gamma_1$ which, inserted in the other equations, leads to

$$\begin{array}{rrrrcl} -\gamma_1 & & -\gamma_3 & -\gamma_5 & = & 0\,, \\ -3\gamma_1 & & & +3\gamma_5 & = & 0\,, \\ 2\gamma_1 & -\gamma_2 & & -\gamma_5 & = & 0\,, \\ 13\gamma_1 & & & -13\gamma_5 & = & 0\,. \end{array} \tag{9}$$

This implies that $\gamma_5 = \gamma_1$ which, inserted in the other equations, leads to

$$\begin{array}{rrrcl} -2\gamma_1 & & -\gamma_3 & = & 0\,, \\ \gamma_1 & -\gamma_2 & & = & 0\,, \end{array} \tag{10}$$

i.e., $\gamma_2 = \gamma_1$ and $\gamma_3 = -2\gamma_1$. In summary, we have $\gamma_2 = \gamma_1$, $\gamma_3 = -2\gamma_1$, $\gamma_4 = -2\gamma_1$, and $\gamma_5 = \gamma_1$. Choosing $\gamma_1 := -1$, we obtain $\vec{x}_2 = -\vec{x}_3 + 2\vec{x}_4 + 2\vec{y}_2 - \vec{y}_3$ as a linear combination of $\vec{x}_3, \vec{x}_4, \vec{y}_2, \vec{y}_3$. It follows that

$$U + V \;=\; \mathrm{span}\{\vec{x}_2, \vec{x}_3, \vec{x}_4, \vec{y}_2, \vec{y}_3\} \;=\; \mathrm{span}\{\vec{x}_3, \vec{x}_4, \vec{y}_2, \vec{y}_3\}\,. \tag{11}$$

Moreover, our computation shows that if $\gamma_1 = 0$ then the only solution of $\gamma_2\vec{x}_3 + \gamma_3\vec{x}_4 + \gamma_4\vec{y}_2 + \gamma_5\vec{y}_3 = \vec{0}$ is $\gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = 0$. This proves that $\{\vec{x}_3, \vec{x}_4, \vec{y}_2, \vec{y}_3\} \subseteq \mathbb{R}^5$ is linearly independent. Eq. (11) and this linear independence imply that $\{\vec{x}_3, \vec{x}_4, \vec{y}_2, \vec{y}_3\} \subseteq U + V$ is a basis.

(c) Any vector $\vec{x} \in U \cap V$ can be written as a linear combination of $\{\vec{x}_2, \vec{x}_3, \vec{x}_4\}$ and of $\{\vec{y}_2, \vec{y}_3\}$, i.e., there are numbers $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5 \in \mathbb{R}$ such that $\vec{x} = \eta_1\vec{x}_2 + \eta_2\vec{x}_3 + \eta_3\vec{x}_4$ and $\vec{x} = -\eta_4\vec{y}_2 - \eta_5\vec{y}_3$, and hence also $\eta_1\vec{x}_2 + \eta_2\vec{x}_3 + \eta_3\vec{x}_4 + \eta_4\vec{y}_2 + \eta_5\vec{y}_3 \;=\; \vec{0}$. We already determined all solutions of this system in (b), namely,

$$\begin{aligned} &\big\{(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) \in \mathbb{R}^5 \;\big|\eta_1\vec{x}_2 + \eta_2\vec{x}_3 + \eta_3\vec{x}_4 + \eta_4\vec{y}_2 + \eta_5\vec{y}_3 \;=\; \vec{0}\big\} \\ &\qquad = \;\big\{(t,\, t,\, -2t,\, -2t,\, t) \;\big|\; t \in \mathbb{R}\big\}\,. \end{aligned} \tag{12}$$

It follows that $\vec{x} = -\eta_4\vec{y}_2 - \eta_5\vec{y}_3 = t(2\vec{y}_2 - \vec{y}_3)$, for some $t \in \mathbb{R}$. Furthermore, any real multiple of $2\vec{y}_2 - \vec{y}_3$ is in $U \cap V$, and $2\vec{y}_2 - \vec{y}_3 \neq \vec{0}$, since $\vec{y}_2$ and $\vec{y}_3$ are linearly independent. Hence, $\{2\vec{y}_2 - \vec{y}_3\} \subseteq U \cap V$ is a basis and

$$U \cap V \;=\; \mathbb{R} \cdot \begin{pmatrix} 2 \\ 0 \\ 3 \\ -1 \\ 1 \end{pmatrix} \;=\; \mathbb{R} \cdot \vec{x}_1\,. \tag{13}$$

Problem 1.2 (6 Points): Let

$$A \;:=\; \begin{pmatrix} 2 & 7 & 9 \\ 1 & 2 & 3 \\ 2 & 5 & 1 \end{pmatrix} \quad \text{and} \quad T \;:=\; \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

be two real 3×3 matrices.

(a) Calculate the determinant of A and decide whether A is invertible or not. If A is invertible then compute the matrix inverse of A.

(b) Calculate the determinant of T and decide whether T is invertible or not. If T is invertible then compute the matrix inverse of T.

Solution.

(a) We first compute the determinant of T using Sarrus' rule, which yields

$$\begin{aligned} \det(A) \;&:= 2 \cdot 2 \cdot 1 + 7 \cdot 3 \cdot 2 + 9 \cdot 1 \cdot 5 - 9 \cdot 2 \cdot 2 - 2 \cdot 3 \cdot 5 - 7 \cdot 1 \cdot 1 \\ &= 4 + 42 + 45 - 36 - 30 - 7 \;=\; 18\,. \end{aligned} \tag{14}$$

Since $\det(A) \neq 0$, the matrix A is invertible. There are several ways to calculate the inverse; we use the matrix of the minors.

$$\alpha_{1,1} := \det \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix} = 2 \cdot 1 - 3 \cdot 5 = -13\,, \tag{15}$$

$$\alpha_{1,2} := \det \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} = 1 \cdot 1 - 3 \cdot 2 = -5\,, \tag{16}$$

$$\alpha_{1,3} := \det \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = 1 \cdot 5 - 2 \cdot 2 = 1\,, \tag{17}$$

$$\alpha_{2,1} := \det \begin{pmatrix} 7 & 9 \\ 5 & 1 \end{pmatrix} = 7 \cdot 1 - 9 \cdot 5 = -38\,, \tag{18}$$

$$\alpha_{2,2} := \det \begin{pmatrix} 2 & 9 \\ 2 & 1 \end{pmatrix} = 2 \cdot 1 - 9 \cdot 2 = -16\,, \tag{19}$$

$$\alpha_{2,3} := \det \begin{pmatrix} 2 & 7 \\ 2 & 5 \end{pmatrix} = 2 \cdot 5 - 7 \cdot 2 = -4\,, \tag{20}$$

$$\alpha_{3,1} := \det \begin{pmatrix} 7 & 9 \\ 2 & 3 \end{pmatrix} = 7 \cdot 3 - 9 \cdot 2 = 3\,, \tag{21}$$

$$\alpha_{3,2} := \det \begin{pmatrix} 2 & 9 \\ 1 & 3 \end{pmatrix} = 2 \cdot 3 - 9 \cdot 1 = -3\,, \tag{22}$$

$$\alpha_{3,3} := \det \begin{pmatrix} 2 & 7 \\ 1 & 2 \end{pmatrix} = 2 \cdot 2 - 7 \cdot 1 = -3\,. \tag{23}$$

The inverse of A is now given as $\left(A^{-1}\right)_{k,\ell} = [\det(A)]^{-1}(-1)^{k+\ell}\alpha_{\ell,k}$, which implies that

$$A^{-1} = \frac{1}{18} \begin{pmatrix} -13 & 38 & 3 \\ 5 & -16 & 3 \\ 1 & 4 & -3 \end{pmatrix}\,. \tag{24}$$

(b) We first compute the determinant of T using Sarrus' rule, which yields

$$\begin{aligned} \det(T) &= 1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 - 3 \cdot 5 \cdot 7 - 1 \cdot 6 \cdot 8 - 2 \cdot 4 \cdot 9 \\ &= 45 + 84 + 96 - 105 - 48 - 72 = 0\,. \end{aligned} \tag{25}$$

Thus T is not invertible.