Intersections of Amoebas

Definition of Amoebas

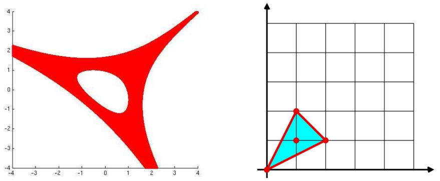
Let $I \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$ with variety $\mathcal{V}(I) \subset (\mathbb{C}^*)^n$. Define:

$$\mathrm{Log}\,|\cdot| : (\mathbb{C}^*)^n \to \mathbb{R}^n, \quad (|z_1| \cdot e^{i \cdot \phi_1}, \ldots, |z_n| \cdot e^{i \cdot \phi_n}) \mapsto (\log|z_1|, \ldots, \log|z_n|)$$

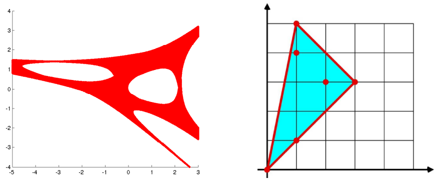
The AMOEBA $\mathcal{A}(I)$ of I is the image of $\mathcal{V}(I)$ under the $\mathrm{Log}\,|\cdot|$-map.

- $\mathcal{A}(f) := \mathcal{A}(\langle f \rangle)$ is a closed set with convex components $E_{\alpha(j)}(f)$ of the complement.
- Each component $E_{\alpha(j)}(f)$ of the complement of $\mathcal{A}(f)$ corresponds to a unique lattice point $\alpha(j)$ in $\mathrm{New}(f)$ via the ORDER MAP.

$f = z_1^2 z_2 + z_1 z_2^2 - 4 \cdot z_1 z_2 + 1$



$f = z_1^3 z_2^3 - 9 z_1^2 z_2^3 + z_1 z_2^5 - 4 z_1 z_2^4 - 4 z_1 z_2 + 1$



Amoebas have countless applications in subjects like complex analysis, real algebraic geometry, dimers and crystal shapes, the geometry of polynomials, and tropical geometry.

Intersections of Amoebas

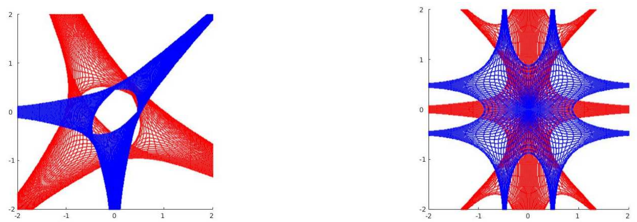
FACTS:

- Amoebas of hypersurfaces have been studied intensively during the last years.
- We know almost nothing about amoebas for the non-hypersurface case.

Theorem (Purbhoo '08). *Let $I \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$ be an ideal. Then*

$$\mathcal{A}(I) = \bigcap_{f \in I} \mathcal{A}(f).$$

KEY IDEA: Investigate intersections of amoebas to gain information about the non-hypersurface case.



Observation: Intersections of amoebas behave like polytopes.

Theorem (Juhnke-Kubitzke, dW.). *Let $\mathcal{F} := \{f_1, \ldots f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$ be a generic collection of Laurent polynomials. Let K be a connected component of $\mathcal{I}(\mathcal{F}) := \bigcap_{j=1}^n \mathcal{A}(f_j)$. Then:*

1. *K admits a face lattice.*
2. *Let $V(K)$ be the VERTICES of K. The INTERSECTION POLYTOPE $P_K := \mathrm{conv}(V(K))$ equals $\mathrm{conv}(K)$.*
3. *P_K is simple.*

The Extension of the Order Map

Observation: The order map can be extended to intersections of amoebas:

Definition. Let $\mathcal{F} := \{f_1, \ldots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$ be a generic collection of Laurent polynomials. Let $V(\mathcal{F})$ be the vertices of $\mathcal{I}(\mathcal{F})$. We define the GENERALIZED ORDER MAP $\mathrm{ord}_{\mathcal{F}}$ as

$$\mathrm{ord}_{\mathcal{F}} : V(\mathcal{F}) \to \mathbb{Z}^{n \times n} \qquad \mathbf{v} \mapsto (\mathrm{ord}_1(\mathbf{v}), \cdots, \mathrm{ord}_n(\mathbf{v}))^T$$

Theorem (Juhnke-Kubitzke, dW.). *Let $\mathcal{F} := \{f_1, \ldots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$ be a generic collection of Laurent polynomials. Let K be a connected component of $\mathcal{I}(\mathcal{F})$ and let P_K be the corresponding intersection polytope. Then:*

1. *$\mathrm{ord}_{\mathcal{F}}$ restricted to $V(P_K)$ is injective.*
2. *$\mathrm{ord}_{\mathcal{F}}$ restricted to the vertices of $\mathrm{conv}(V(\mathcal{F}))$ is injective.*

KEY QUESTION: How many connectivity components can $\mathcal{I}(\mathcal{F})$ have?

The Classical Bernstein Theorem

Theorem (Bernstein Theorem '75). *Let*

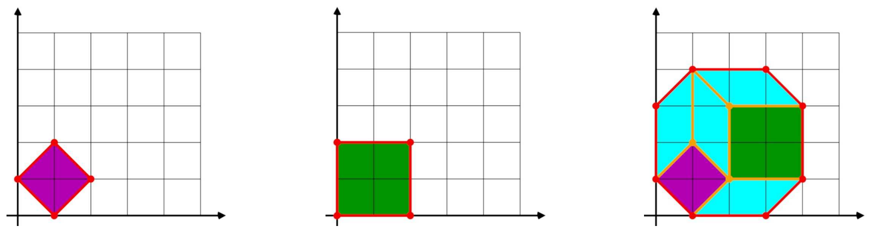
- *$A_1, \ldots, A_n \subset \mathbb{Z}^n$ be finite sets such that their union generates $\mathbb{Z}^n$ as an affine lattice.*
- *$P_i \subseteq \mathbb{R}^n$ be the convex hull of A_i, and*
- *$\mathbb{C}^{A_i}$ be the space of Laurent polynomials in $z_1, \ldots, z_n$ with support set A_i.*

Then there exists a dense Zariski open subset $U \subseteq \mathbb{C}^{A_1} \times \cdots \times \mathbb{C}^{A_n}$ with the following property: For any $(f_1, \ldots, f_n) \in U$, the number of solutions of the system of equations

$$f_1(\mathbf{z}) = \cdots = f_n(\mathbf{z}) = 0$$

in $(\mathbb{C}^*)^n$ equals the mixed volume $\mathrm{MV}(P_1, \ldots, P_n)$.

Geometric interpretation: Mixed subdivisions and mixed cells allow to compute mixed volumes.



Main Result: The Amoeba Bernstein Theorem

Theorem (Juhnke-Kubitzke, dW.). *Let $\mathcal{F} = \{f_1, \ldots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$ be a generic collection of Laurent polynomials. The number of connected components of the intersection $\mathcal{I}(\mathcal{F})$ is bounded from above by the mixed volume $\mathrm{MV}(\mathrm{New}(f_1), \ldots, \mathrm{New}(f_n))$.*

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