Norms of Roots of Trinomials

Trinomials

Investigate TRINOMIALS

$$f = z^{s+t} + pz^t + q, \quad p \in \mathbb{C}, q \in \mathbb{C}^*, \; s,t \in \mathbb{N}^*, \gcd(s,t) = 1.$$

KEY QUESTION: How are the coefficients p, q related to the norms of roots?

⇒ Classical late 19th / early 20th century problem. E.g., Nekrassoff: "Ueber trinomische Gleichungen", 1883.

⇒ Algebraically solved by P. Bohl in "Zur Theorie der trinomischen Gleichungen" in 1908.

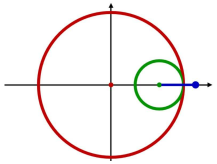
⇒ Geometrical / topological properties of coefficient space of trinomials are UNKNOWN.

Hypotrochoids

For $R > r$, a HYPOTROCHOID with parameters $R, r \in \mathbb{Q}_{>0}, d \in \mathbb{R}_{>0}$ is the parametric curve in $\mathbb{R}^2 \cong \mathbb{C}$ given by

$$(R-r) \cdot e^{i \cdot \phi} + d \cdot e^{i \cdot \left(\frac{r-R}{r}\right) \cdot \phi}, \quad \phi \in [0, 2\pi).$$

Geometrically, it is the trajectory of some fixed point with distance d to the center of a circle with radius r rolling (from the interior) on a circle with radius R.

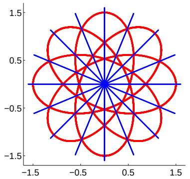
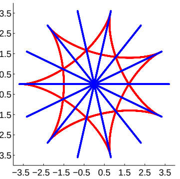
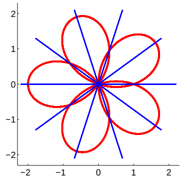


The Geometry of Norms of Trinomials

Theorem (Theobald, dW.). *Let $f = z^{s+t} + pz^t + q$ be a trinomial with $p \in \mathbb{C}$ and $q \in \mathbb{C}^*$.*

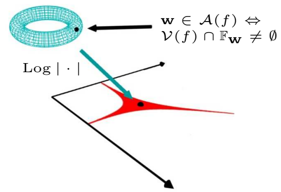
1. *f has a root of norm v if and only if the coefficient p lies on a particular hypotrochoid depending on s, t, q and v.*
2. *There exist two roots of f with identical norm if and only if p is a singular point of the corresponding hypotrochoid. In detail:*
   (a) *f has two distinct roots with identical norm v if and only if p is located on a real double point of the hypotrochoid,*
   (b) *f has a root of multiplicity 2 with norm v if and only if the corresponding hypotrochoid is a hypocycloid and p is a cusp of it, and*
   (c) *f has more than two roots with norm v if and only if $p = 0$ if and only if the hypotrochoid is a rhodonea curve with a multiple point of multiplicity $s+t$ at the origin.*

Example. *Let $f = z^8 + pz^3 + \frac{1}{2}$, $g = z^7 + pz^2 + \frac{5}{2}$, $h = z^5 + pz + 1$. Then f, g, h has a root of norm 1 if and only if $p \in \mathbb{C}$ is located on the hypotrochoids with parameters $(R, r, d) = (8/3, 5/3, 1/2)$, $(7/2, 5/2, 5/2)$, $(5, 4, 1)$.*

Sketch of a Proof for the Geometry of Norms of Trinomials

⇒ A Laurent polynomial f has a root of norm v if and only if $\mathrm{Log}\,|v| = (\log|v_1|, \ldots, \log|v_n|)$ is contained in the AMOEBA $\mathcal{A}(f) := \mathrm{Log}\,|\mathcal{V}(f)|$.

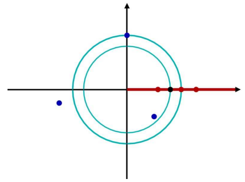


⇒ The amoeba $\mathcal{A}(f)$ and $\mathrm{Log}\,|\cdot|$ come with a fiber bundle $(S^1)^n \to (\mathbb{C}^*)^n \to \mathbb{R}^n$, induced by holomorphic logarithm. This induces a fiber function $f^v : (S^1)^n \to \mathbb{C}$ depending on the coefficients p, q of f.

⇒ In the trinomial case a computation shows that f^v vanishes if and only if:

$$-p \in \left\{ e^{i \cdot 2\pi \arg(q) \cdot s} \cdot \left((R-r) \cdot e^{i \cdot \phi} + d \cdot e^{i \cdot \left(\frac{r-R}{r}\right) \cdot \phi} \right) : \phi \in [0, 2\pi) \right\}.$$

for suitable R, r, d depending on s, t, q, v. This is the parametric description of a hypotrochoid.

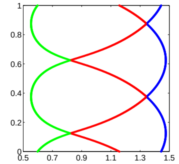


⇒ Bohl's results imply: if $p, q \neq 0$, then at most two roots have the same norms. Singularities correspond to: For a particular choice of p one finds two arguments ϕ_1, ϕ_2 for norm v such that $f(v \cdot e^{i\phi_1}) = f(v \cdot e^{i\phi_2}) = 0$. □

Corollary (Sommerville, 1920). *Let $\gamma : [0, 2\pi) \to \mathbb{C}$, $\phi \mapsto v^s e^{i \cdot s\phi} + |q| v^{-t} e^{i \cdot (\arg(q) - t\phi)}$ be a hypotrochoid. Then the arguments of all singularities of γ are contained in $\{(s \arg(q) + k \cdot \pi)/(s+t) \; : \; k = 1, 2, \ldots, s+t\}$.*

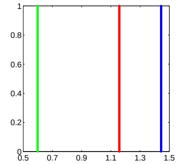
The Topology of Norms of Trinomials

Let T_A denote the space of trinomials with support $A := \{0, t, s+t\}$. For $f \in T_A$ let $v_1^f \leq \cdots \leq v_{s+t}^f \in \mathbb{R}_{>0}$ be the norms of roots of trinomials and $U_j^A := \{f \in T_A \; : \; v_j^f \neq v_{j+1}^f\}$.



QUESTION: Is every set U_j^A connected or simply connected in T_A?

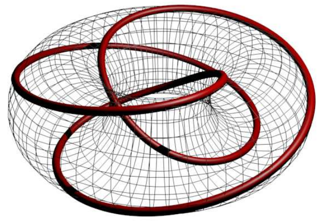
Example. *Let $f = z^3 + 1.5 \cdot e^{i \cdot \arg(p)} z + e^{i \cdot \arg(q)}$.*



AIM: *Construct a path γ in T_A from $(p_1, q_1) = (1.5 \cdot e^{i \cdot \pi/2}, 1)$ to $(p_2, q_2) = (1.5 \cdot e^{-i \cdot \pi/6}, 1)$ such that $\gamma \in U_1^A$.*

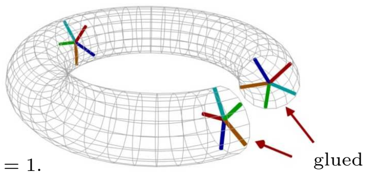
⇒ *Impossible if* $\arg(q) = 0$ *for every point on* γ.

⇒ *Possible along* $\gamma : [0,1] \to T_A, k \mapsto (1.5 \cdot e^{i(1/4 + 2k/3) \cdot 2\pi}, e^{i \cdot 2k\pi})$.



Theorem (Theobald, dW.). *For all $j \neq t$ the sets $U_j^A \subset T_A$ and their complements can be deformation retracted to a $K(s+t, s)$ torus knot. In particular, they are connected, but not simply connected.*

Figure on the right: The situation in T_A restricted to $|q| = 1$.



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see ArXiv 1411.6552.