# Connection between singular integrals & functional calculi

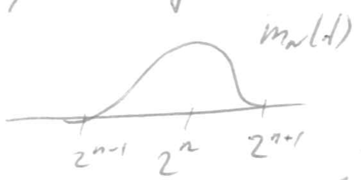
On the one hand, it's well known that bounded fcts $m \in L^\infty$ (spectral multipliers) of self-adjoint operators $A$ (in $L^2(\mathbb{R}^n)$, for example) are $L^2$-bdd, i.e.,

given $A : D(A) \to \mathcal{H}$ densely defined and s.a., then $m(A)_n \equiv \int_{\sigma(A)} m(\lambda)\, dE_A(\lambda)$ is $L^2$-bdd. This is the content of the spectral thm. A special instance of this assertion is known under Plancherel's thm whenever $A$ is $-i\nabla$, i.e., $m$ is a Fourier multiplier. Then $m(D)f(x)$ is defined as $\int \left( \int m(\xi - \eta) \hat{f}(\eta)\, d\eta \right) e^{2\pi i x \cdot \xi}\, d\xi$ and we have $\|m(D)f\|_{L^2} = \|m(\cdot)\|_{L^\infty} \|f\|_2$.

A much more subtle question is to find sufficient condition for $L^p$-bddness of such spectral multipliers $m$.

On the one hand there is the Hörmander-Mikhlin thm which asserts that $m(A)$ is $L^p$-bdd when $m$ is sufficiently smooth in the sense $m \in H^s_{loc}(\mathbb{R})$, $s > s(d, A)$. For Fourier multipliers, this is well known (up to the optimal $s(d,A)$ at least, which is a very hard problem), whereas the situation for ~~arbitrar~~ more general $A$ is still widely open and an active field of research. (Consider, e.g. $-\Delta - V$ or general $T(-i\nabla) - V$ or $T(-i\nabla)|_M \ldots$)

Such spectral multiplier ~~problems~~ theorems are invaluable tools in the study of (nonlinear) PDE and applied mathematics in general as they allow to decompose problems on relevant length scales and study these problems in a separated way. E.g. $m(\lambda) \equiv m_N(\lambda) = \Phi\left(\frac{\lambda}{2^n}\right) - \Phi\left(\frac{\lambda}{2^{n+1}}\right)$ where $\Phi \in C_c^\infty$ at $0$




Thus the Hörmander-Mikhlin thm provides us with a functional calculus for $L^p$ in ~~the same~~ a similar way as the spectral thm in $L^2$.

[For $\Psi$DO's this goes under a theorem of Calderón and Vaillancourt giving $L^p$-, $H^s$, $1$-bddness of $\Psi$DO's whose symbols belong, e.g., to $S^m$]

However, there are situations where it's not possible or desirable to take smooth projections and one needs to deal with rough cut-offs such as indicator fcts. Recall $\widehat{H} = \mathrm{sgn}(\xi) \Rightarrow$ projection onto half-plane $\xi_1 \geq 0$ can be

write as $\frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}\,\xi_1 = \begin{cases} 1 & \xi_1 \geq 0 \\ 0 & \xi_1 < 0 \end{cases}$

or onto $\xi_1 \in [0,1] \to \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}\,\xi_1 - \underbrace{\left(\frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(\xi_1 - 1)\right)}_{\begin{cases} 1 & \xi_1 > 1 \\ 0 & \xi_1 < 1 \end{cases}}$

$= \frac{1}{2}\left(\mathrm{sgn}\,\xi_1 - \mathrm{sgn}(\xi_1 - 1)\right)$

$\to$ projection onto cube $(\xi_1, \xi_2) \in (0,1)^2$

$\frac{1}{2}\left(\mathrm{sgn}\,\xi_1 - \mathrm{sgn}(\xi_1 - 1)\right) \circ \frac{1}{2}\left(\mathrm{sgn}\,\xi_2 - \mathrm{sgn}(\xi_2 - 1)\right)$

These operators ~~are just~~ arise as singular integral operators (here convolution with p.v. $1/x$) and so it's (like in the previous, smooth case) inevitable to find "efficient" sufficient criteria for their $L^p$-bddness as well.

($\to$ consider also Krantz ~~for~~ in view of failure of $L^p$-bddness of the disc multiplier)

$L^\infty \supset \int_B |f(x) - f_B|\,dx \lesssim 1 \;\forall B$ $\quad T: L^\infty \to BMO$, $\; T: H^1 \to H^1 \subseteq L^1$

$H^1$ and $BMO$ are dual to each other

Unsatisfactory $L^\infty$ theory $\to BMO$
$L^1$ theory $\to H^1$

weighted analogs $\to$ Muckenhaupt weights $A_p$, especially $A_\infty$ $(A_p \subseteq A_{p'}\ \ p < q)$

$w \in A_p \Leftrightarrow \frac{1}{|B|}\int_B w(x)\,dx \cdot \left[\frac{1}{|B|}\int_B w(x)^{-p'/p}\right]^{p/p'} \lesssim 1$

$\|Mf\|_{L^p(w\,dx)} \lesssim \|f\|_{L^p(w\,dx)}$

## 3 Singular integrals → will lead to a functional calculus in $x$-space ($FT \Rightarrow$ fun. calculus in $\xi$-space ΨDOs in phase space) (1)

## 3.1 Integrals of Marcinkiewicz

↗ express the fact that a general point of a measurable set is ~~almost~~ completely surrounded by other points of the set


Let $F \subseteq \mathbb{R}^d$ be a closed set, $\delta(x) = dist(x, F)$

$$I(x) := \int_{|y|\le 1} \frac{\delta(x+y)}{|y|^{d+1}}\, dy = \int_{|y-x|\le 1} \frac{\delta(y)}{|x-y|^{d+1}}\, dy$$

Theorem 3.1 a) When $x \in F^c$, then $I(x) = \infty$

b) For almost all $x \in F$, we have $I(x) < \infty$.

Conclusion a) is evident since $F^c$ is an open set, i.e., if $x \in F^c$ then $\delta(x+y) \ge c > 0$ whenever $y$ sits in a tiny neighborhood of the origin.

The conclusion b) is of interest here and it states in effect that $\delta(x+y) = o(|y|)$ in a somewhat refined way, at least in average → leads to the convergence of $I(x)$.

The theorem will be a simple consequence of the following lemma which is a more quantitative expression of the same fact.

Lemma 3.2 Let $F \subseteq \mathbb{R}^d$ be closed with $|F^c| < \infty$ and let $I_*(x) = \int_{\mathbb{R}^d} \frac{\delta(x+y)}{|y|^{d+1}}\, dy$

Then $I_*(x) < \infty$ for a.e. $x \in F$ and moreover

$$\int_F I_*(x)\, dx \le c\, |F^c|. \quad (*)$$

1) then Hilbert + Riesz transform as in Grafakos p. 314–317, 324–329

2) Stein, Chapter II

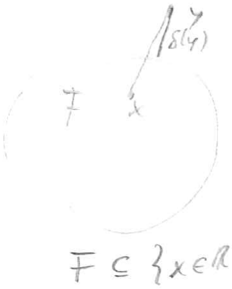
Proof Since $I_*(x) \ge 0$, it suffices to prove $(*)$.

$$\int_F I_*(x)\,dx = \int_F dx \int_{\mathbb{R}^d} dy\, \frac{\delta(y)}{|x-y|^{d+1}} \underset{\delta(y)=0 \text{ for } y\in F}{=} \int_F dx \int_{{}^cF} dy\, \frac{\delta(y)}{|x-y|^{d+1}}$$

$$= \int_{{}^cF} dy\, \delta(y) \underbrace{\int_F \frac{dx}{|x-y|^{d+1}}}_{\substack{\le \\ y \in {}^cF}\ \int_{|x| \ge \delta(y)} \frac{dx}{|x|^{d+1}} \ \lesssim\ \frac{1}{\delta(y)}} \lesssim \int_{{}^cF} dy = |{}^cF|$$

$\boxed{\phantom{x}}$

$|x-y| > \delta(y)$ as $x$ varies over $F$




$F \subseteq \{x \in \mathbb{R}^d : |x-y| > \delta(y)\}$

$$\Rightarrow \int_F \frac{dx}{|x-y|^{d+1}} \le \int_{|x-y|>\delta(y)} \frac{dx}{|x-y|^{d+1}} = \int_{|x|>\delta(y)} \frac{dx}{|x|^{d+1}}$$

Proof of Thm 3.1 b), i.e., $\int_{|y|<1} \frac{\delta(x+y)}{|y|^{d+1}} < \infty$ for $x \in F$

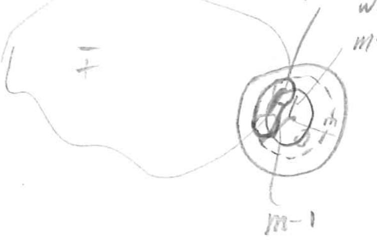
Let $F_m \equiv F \cup B_0(m)^c$ and assume $m > 0$ so large that $F \cap B_0(m-2) \neq \emptyset$ which is a closed set whose complement, $F^c \cap B_0(m)$, has finite measure $\to$ we can apply Lemma 3.1 to $F_m$; so let $\delta_m(x) = dist(x, F_m)$ and observe that $\delta_m(x+y) = \delta(x+y)$ whenever $|y| < 1$ and $x \in B_0(m-2) \cap F$ $\Rightarrow x+y \in B_0(m-1)$

here could $x+y$ live when $x \in F \cap B_0(m-2)$

$\Rightarrow I(x) < \infty$ ~~whenever~~ for a.e. $x \in F \cap B_0(m-2) \subseteq F_m$

(we're not using the quantitative bound here)

$\to$ letting $m \to \infty$ gives the desired result ☒




only non-trivial when $x+y \notin F$

$dist(x+y, F)$

$|x+y| \leq m-1 \Rightarrow dist(x+y, B_0(m-2)^c) \geq 1$

Another variat $I^{(\lambda)}(x) = \int_{|y|\leq 1} \frac{\delta^\lambda(x+y)}{|y|^{d+\lambda}} dy$, $\lambda > 0$; similarly, $I_*(x)$ can be replaced by

$$I_*^{(\lambda)}(x) = \int_{\mathbb{R}^d} \frac{\delta^\lambda(x+y)}{|y|^{d+\lambda}}, \lambda > 0$$

and one obtains similar conclusions with the above methods.

## 3.2 Introductory example – ~~the~~ Hilbert and Riesz transforms



The Hilbert transform is the prototype of all singular transform and has deep connections related to summability problems in Fourier analysis (aka Bochner-Riesz)

Historically, the Hilbert transform depended essentially on complex analysis where it was tied to the theory of harmonic functions on the upper half space (recall that the Poisson kernel solved $(-\partial_t^2 - \Delta_x) e^{-t|p|} = 0$ in $(t,x) \in ([0,\infty) \times \mathbb{R}^d)$ ?)

However with the development of the Chicago school around Zygmund and his doctoral student Calderón, and the extension of one-dimensional theory to higher dimensions, real-variable methods slowly replaced complex analysis. The higher-dimensional framework proved flexible enough for generalizations and led to the introduction of singular integrals in other areas of mathematics.

Nowadays, singular integrals are intimately connected ~~to~~ with PDE, operator theory and other fields.

Examples: ∘ fractional Laplacian $\langle u, |p|^\alpha v\rangle = \int \frac{\overline{(u(x)-u(y))}(v(x)-v(y))}{|x-y|^{d+\alpha}}$

∘ projection operators, such as in the Brown-Ravenhall model of relativistic quantum mechanics. Consider the Dirac operator with an external ~~mag~~ electric field $V$, i.e. $D_V \equiv \underbrace{-i\vec{\alpha}c\vec{\nabla} + \beta \cancel{mc^2} m}_{D_0} + V$ in $L^2(\mathbb{R}^3 : \mathbb{C}^4)$

$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$

For $V$ such that $(D_0 - z)^{-1} - (D_V - z)^{-1} \in \gamma^\infty$ for $z \in \rho(D_V) \cap \rho(D_0)$, we have $\sigma_{ess}(D_V) = \sigma(D_0) = (-\infty, -mc^2] \cup [mc^2, \infty)$ and $\sigma_{disc} \subseteq (-mc^2, mc^2)$

To consider only "electrons" with respect to $D_0$, one projects onto $\Lambda_0^+ L^2(\mathbb{R}^3 : \mathbb{C}^4)$ where $\Lambda_0^\pm = \frac{1}{2}\left(1 \pm \frac{D_0}{|D_0|}\right)$

By a unitary transform of Foldy and Wouthuysen, one can map $D_V$ to a s.a. operator on $L^2(\mathbb{R}^3 : \mathbb{C}^2)$ where it then reads

$$\sqrt{-c^2\Delta + c^4} + \begin{pmatrix} \Phi_0(p/c) \\ \Phi_1(p/c) \end{pmatrix} V \otimes \mathbb{1}_{\mathbb{C}^4} \begin{pmatrix} \Phi_0(p/c) \\ \Phi_1(p/c) \end{pmatrix}$$ where $\Phi_j : L^2(\mathbb{R}^3 : \mathbb{C}^2) \to L^2(\mathbb{R}^3 : \mathbb{C}^4)$

$$\Phi_0(p) = \phi_0(p) \otimes \mathbb{1}_{\mathbb{C}^2} \quad \text{and} \quad \phi_j(p) = \sqrt{\frac{\sqrt{p^2+1} + (-1)^j}{2\sqrt{p^2+1}}}$$

$$\Phi_1(p) = \phi_1(p) \cdot \frac{\vec{\sigma}\vec{p}}{|p|}$$

frequently, commutators with $C_c^\infty$ or $\mathcal{S}$ such as $[\chi, \phi_j]$ appear and need to be controlled → better machinery later → ΨDOs

One can compute, e.g. $\frac{\vec{\sigma}\vec{p}}{|p|}(x,y) = \frac{\vec{\sigma}(\vec{x}-\vec{y})}{|x-y|^4}$ (∼ Riesz) and $\frac{\vec{\sigma}\vec{p}}{\sqrt{p^2+m^2}} \cong \frac{K_2(m|x-y|)\,\vec{\sigma}(\vec{x}-\vec{y})}{|x-y|^2}$

$\left\{ \begin{array}{ll} (m|x-y|)^{-2} & m|x-y| \ll 1 \\ \frac{e^{-m|x-y|}}{\sqrt{m|x-y|}} & m|x-y| \gg 1 \end{array} \right.$

# Definition and basic properties of the Hilbert transform

There are several equivalent ways to introduce the Hilbert transform; here, we define it as a convolution operator with a certain principal value distribution and see later other equivalent definitions.

Let $W_0 \in \mathcal{S}'(\mathbb{R}^1)$ be defined as $\langle W_0, \varphi\rangle_{\mathcal{S},\mathcal{S}'} = \frac{1}{\pi}\lim\limits_{\epsilon\to 0}\int\limits_{\epsilon\le|x|\le 1}\frac{\varphi(x)}{x}\,dx + \frac{1}{\pi}\int\limits_{|x|\ge 1}\frac{\varphi(x)}{x}\,dx$,

for all $\varphi\in\mathcal{S}(\mathbb{R}^1)$. Since $\int\limits_{\epsilon\le|x|\le 1}\frac{dx}{x} = 0$, we have

$$|\langle W_0,\varphi\rangle| \le \frac{1}{\pi}\lim_{\epsilon\to 0}\underbrace{\int\limits_{\epsilon<|x|<1}\frac{|\varphi(x)-\varphi(0)|}{|x|}}_{\le \int\limits_{\epsilon<|x|<1}\frac{|\nabla\varphi|\,|x|}{|x|} \le 2\sup_x|\nabla\varphi(x)|} + \frac{2}{\pi}\sup_{x\in\mathbb{R}}|x\cdot\varphi(x)| \le \frac{2}{\pi}\left(\|\nabla\varphi\|_\infty + \|x\varphi\|_\infty\right)$$

$\Rightarrow W_0\in\mathcal{S}'(\mathbb{R})$

**Definition 3.3** The truncated Hilbert transform at height $\epsilon>0$ of a $f\in L^p(\mathbb{R})$, $1\le p<\infty$, is given by $H^{(\epsilon)}(f) = \frac{1}{\pi}\int\limits_{|y|>\epsilon}\frac{f(x-y)}{y}\,dy = \frac{1}{\pi}\int\limits_{|x-y|>\epsilon}\frac{f(y)}{x-y}\,dy$ (Goal $H: L^p\to L^p$-bdd)

The Hilbert transform of $\varphi\in\mathcal{S}(\mathbb{R})$ is defined by
$H(\varphi)(x) = (W_0 * \varphi)(x) = \lim\limits_{\epsilon\to 0} H^{(\epsilon)}(\varphi)(x)$

Observe that $H^{(\epsilon)}(f)$ is well-defined for all $1\le p<\infty$ since $x^{-1}\in L^{p'}(|x|>\epsilon)$ for any $p'>1$ and Hölder

For $\varphi\in\mathcal{S}$, the integral $\int\limits_{\mathbb{R}}\frac{\varphi(y)}{x-y}\,dy$ may fail to converge absolutely for any $x\in\mathbb{R}$; instead, it is defined as the limit of the absolutely convergent integrals

$$\int\limits_{|y-x|>\epsilon}\frac{\varphi(y)}{|x-y|}\,dy \quad \text{as } \epsilon\to 0.$$

Such limits are called principal values (p.v.) integrals. In this notation,

$$(H\varphi)(x) = \frac{1}{\pi}\,\mathrm{p.v.}\int\limits_{\mathbb{R}}\frac{\varphi(y)}{x-y}\,dy$$

Remark 3.4 We extend the definition of the Hilbert transform to a bigger class of functions. At the beginning of this section, we saw that we merely needed, in fact, to require Hölder continuity of $f_n$ to treat the singular part, i.e., $|f(x)-f(y)| \leq c_x |x-y|^{\epsilon_x}$ for some $c_x, \epsilon_x > 0$ at any point $x \in \mathbb{R}$, whenever $|x-y| < \delta_x$.

$$\int_{\epsilon<|x-y|<\delta_x} \frac{dy}{x-y} = 0$$

$$\rightarrow H^{\epsilon}(f)(x) = \frac{1}{\pi} \int_{\epsilon<|x-y|<\delta_x} \frac{f(y)-f(x)}{x-y} + \frac{1}{\pi} \int_{|x-y|>\delta_x} \frac{f(y)}{x-y} dy$$ and both integrals converge absolutely.

Exercise Compute $H(\mathbb{1}_{[a,b]})(x) = \frac{1}{\pi} \log \frac{|x-a|}{|x-b|}$ by distinguishing between $0<x-b$, $x-a<0$, $x-b<0<x-a$

($\Rightarrow H$ neither $L^1$- nor $L^\infty$-bdd!)

Alternative characterization of Hilbert transform via Fourier transform & connection to plane multipliers (especially Fefferman, (1971), but see in particular Stein 1970, p. 99

(In the exercises: $Q_t \xrightarrow{t\to 0} H$ where $Q_t = \mathrm{sgn}(\xi)\, e^{-2\pi t|\xi|}$, i.e., $Q_t * f \to Hf$ in $\mathcal{S}'$-sense)

Let $\varphi \in \mathcal{S}(\mathbb{R})$, then $\langle \widehat{W_0}, \varphi \rangle_\xi = \langle W_0, \hat\varphi \rangle := \frac{1}{\pi} \lim_{\epsilon\to 0} \int_{|\xi|>0} \hat\varphi(\xi) \frac{d\xi}{\xi} = \frac{1}{\pi} \lim_{\epsilon\to 0} \int_{\frac{1}{\epsilon}>|\xi|>\epsilon} \frac{d\xi}{\xi} \int_{\mathbb{R}} dx\, e^{-2\pi i x\xi} \varphi(x)$

($H(\varphi) = W_0 * \varphi = pv\frac{1}{x} * \varphi$)

$$= \lim_{\epsilon\to 0} \int dx\, \varphi(x) \Big[ \underbrace{\frac{1}{\pi} \int_{\frac{1}{\epsilon}>|\xi|>\epsilon} \frac{d\xi}{\xi} e^{-2\pi i x\xi}}_{\text{asymmetric} \times \text{symmetric} \to 0,\ \times \text{ asymmetric} \neq 0} \Big]$$

$$= \lim_{\epsilon\to 0} \int dx\, \varphi(x) \Big[ \frac{-i}{\pi} \int_{\frac{1}{\epsilon}>|\xi|>\epsilon} \frac{d\xi}{\xi} \sin(2\pi x\xi) \Big]$$

$$= \lim_{\epsilon\to 0} \int dx\, \varphi(x) \Big[ \Big(\frac{-i}{\pi} \mathrm{sgn}\, x\Big) \underbrace{\int_{\frac{1}{2\pi\epsilon}>|\xi|>\frac{\epsilon}{2\pi}} \frac{d\xi}{\xi} \sin(|x|\cdot\xi)}_{\text{uniformly bdd by 8 (exercise)}} \Big]$$

and converge to $2\cdot\frac{\pi}{2}$ as $\epsilon\to 0$ whenever $x\neq 0$
$\Rightarrow$ by dominated convergence, we can pull $\lim_{\epsilon\to 0}$ inside

$$= \int dx\, \varphi(x) \cdot (-i\,\mathrm{sgn}\, x)\, dx, \text{ i.e., } \widehat{W_0} = -i\,\mathrm{sgn}\,\xi$$

~~Further properties:~~

$\Rightarrow \quad (Hf)(x) = \left(\hat{f}(\xi)\,(-i\,\mathrm{sgn}\,\xi)\right)^{\vee}(x)$ .

Further properties (exercise)

$\|Hf\|_2 = \|f\|_2$ , $H^2 = H \cdot H = -\mathbb{1}$ , $H^* = -H$

Of interest for us is that $H$ is $L^2$-bdd and that $\int \frac{1}{x}\,dx = 0$ in p.v.

and $\hat{H}(\xi) \in L^\infty$

and $\left|\nabla \frac{1}{x}\right| = \frac{1}{|x|^2}$

## 3.3 Review of certain aspects of harmonic analysis

Goal: $L^p$-bddness of $H$ and $L^p$-convergence / a.e.-convergence of $H^{(\varepsilon)}$.

Let us single out some features of the Hilbert transform so that in terms of their aspects we can describe the d-dimensional singular integrals treated here

a) $L^2$ theory: We're dealing here, as in the general case, with an operator that ~~translates~~ commutes with translations $\Rightarrow$ tools of convolution, FT (Plancherel) in $\mathbb{R}^d$ are unavoidable

b) $L^p$ theory: we'll show that $H$ is $L^p$ bdd; in case of the Hilbert transform, this classical thm was proved by M. Riesz using complex analysis $\to$ inappropriate in the general context and we'll obtain it as a consequence of interpolating between the $L^2$ and the $L^1$ theory.

c) $L^1$ theory: $H$ is not $L^1$-bdd, but at least of weak type $(1,1)$. $\to$ similar in general context. Historically, this was first proved by Besicovitch and Titchmarsh for the Hilbert transform and then generalized by Calderón and Zygmund using the real-variable theory outlined in the first two chapters.

d) Special properties of the Hilbert transf.: (i) commutes with translations, but also with dilations $x \mapsto \delta x$ $(\delta > 0)$ $\to$ theorems describing d-dim generalizations are essentially invariant under dilations

(ii) connection with analytic functions

[What about invariance under frequency modulations?]

The structure of translation invariant transformations that are bounded on $L^1$ or $L^2$ is both simple and well understood.

Recall $C_0(\mathbb{R}^d)$... space of functions ~~tan~~ vanishing at $\infty$ and its dual space
$\mathcal{B}(\mathbb{R}^d)$... can be identified with the Banach space of all finite measures $d\mu$ with norm $\|d\mu\| = \int_{\mathbb{R}^d} d\mu$

$L^1(\mathbb{R}^d)$ can therefore be identified as a subspace of $\mathcal{B}(\mathbb{R}^d)$ by the isometry

$\cancel{f \mapsto}$ $f(x) \mapsto f(x)dx$ where $dx$... Lebesgue measure

~~We study the clas~~

Basic operations: convolution and FT

If $\mu_1, \mu_2 \in \mathcal{B}(\mathbb{R}^d)$, then $\mu = \mu_1 * \mu_2$ is defined by $\mu(f) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x+y)\, d\mu_1(x)\, d\mu_2(y)$

$\rightarrow \mu_1 * \mu_2 = \mu_2 * \mu_1$ and $\|\mu\| \le \|\mu_1\| \|\mu_2\|$

If $f \in L^1$, then $g := f * \mu = \int f(x-y)\, d\mu(y) \in L^1$ with $\|g\|_1 \le \|f\|_1 \|\mu\|$

and similarly if $f \in L^p$, then $\|g\|_p \le \|f\|_p \|\mu\|$, i.e., the transformation

$$f \mapsto \int f(x-y)\, d\mu(y), \quad \mu \in \mathcal{B}(\mathbb{R}^d)$$

is bdd in $L^p$ and, in fact, commutes with translations $x \mapsto x+h$. This class of transformations is characterized in the following

<u>Proposition 3.5</u> Let $T$ be a bdd linear transf mapping $L^1$ to itself. ~~Then~~
$\Rightarrow$ $T$ commutes with translations $\Leftrightarrow$ $\exists$ a measure $\mu \in \mathcal{B}(\mathbb{R}^d)$ s.t.

~~$(Tf)(x) = (T f_h)(x+h)$~~ $\qquad T(f) = f * \mu, \quad f \in L^1(\mathbb{R}^d)$

$(Tf)(x+h) = (T f(\cdot - h))(x)$

$(Tf)(x) = \int \mu(x-y) f(y+h)\, dy$
$= (Tf)(x-h)$

In this case, one has $\|T\|_{L^1 \to L^1} = \|\mu\|$

FT.: For $\mu \in \mathcal{B}(\mathbb{R}^d)$ we have $\hat{\mu}(\xi) = \int e^{2\pi i x \cdot \xi}\, d\mu(x)$, in particular, FT is defined for all $f \in L^1$ with $\hat{f} \in C_0$ (Riemann-Lebesgue); if $\mu = \mu_1 * \mu_2 \Rightarrow \hat{\mu} = \hat{\mu}_1 \cdot \hat{\mu}_2$; if $f \in L^1 \cap L^2 \Rightarrow \hat{f} \in L^2$, $\|\hat{f}\|_2 = \|f\|_2$ and $\mathcal{F}$ can be extended by continuity to all of $L^2$ s.t. $\mathcal{F}$ is unitary on $L^2$. Moreover, by continuity, if $g = f * \mu$ with $f \in L^2$, $\mu \in \mathcal{B}(\mathbb{R}^d)$, we have $\hat{g} = \hat{f} \cdot \hat{\mu}$.

The $L^2$ analog of Prop 3.5 is the following

**Proposition 3.6** Let $T$ be a $L^2$-bdd linear transformation

$\Rightarrow$ $T$ commutes with translations $\Leftrightarrow$ $\exists$ an ~~linear~~ multiplier $m \in L^\infty(\mathbb{R}^d)$ s.t.

$$(Tf)^\wedge(\xi) = m(\xi)\hat{f}(\xi), \quad f \in L^2 \quad \text{and} \quad \|T\|_{L^2 \to L^2} = \|m\|_{L^\infty}.$$

$\Rightarrow$ Study of translation invariant ops on $L^p$ is much harder and still not completely resolved (Bochner–Riesz etc)

However, for an important class of tranformations much has been done, namely ~~singular~~ convolution operators with a (non-oscillatory) singular kernel, each having its only singularities at a finite point (e.g., the origin), and at infinity.

(singularities on manifolds much harder...)

## 3.4 The heart of the matter

We present a first, preliminary result that we will generalize a bit later (concerning the cancellation condition) and discuss some of its drawbacks (the a priori $L^p$, e.g. $p=2$, boundedness for some $p > 1$ e.g. $p=2$)

**Theorem 3.7** Let $K \in L^2(\mathbb{R}^d)$ and suppose

a) $|\hat{K}(\xi)| \leq B$ ($\to$ $L^2$-bddness assumed a priori) (too strong cancellation condition $\to$ Hörmander later) see also chapter on almost orthogonality!

b) $K \in C^1$ outside the origin and $|\nabla K(x)| \leq \frac{B}{|x|^{d+1}}$.

For $f \in L^1 \cap L^p$ we set $\quad (Tf)(x) = \int K(x-y) f(y)\, dy.$

$\Rightarrow$ There exists a constant $A_p$ so that $\|Tf\|_p \leq A_p \|f\|_p$ and one can extend $T$ to all of $L^p$ by continuity. The constant $A_p = A(p, B, d)$ and does not depend on the $L^2$ norm of $K$.

Remark: The assumption $K \in L^2$ is only made for the purpose of having a direct definition of $Tf$ on a dense subset of $L^p$ (here $L^1 \cap L^p$) and could be replaced by weaker assumptions such as ~~the~~ $K \in L^1 + L^2$ (or $K \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ mapping $K: \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$, $\mathcal{D} = C_c^\infty$, $\mathcal{D}'$ dual)

In the applications, the hypothesis $K \in L^2$ is of no consequence ~~bt~~ as it can be dispended ~~by~~ with by an appropriate limiting process, since the final bounds in Thm 3.7 do not depend on $\|K\|_{L^2}$, see later Thm 3.8.

<u>Proof of Thm 3.7</u>

To prove the $L^p$-bddness it suffices to prove that is of strong-type $(2,2)$ and of weak-type $(1,1)$ $\Rightarrow$ by Marcinkiewicz this gives that $T$ is of strong-type $(p,p)$ for $1<p<2$. For $2<p<\infty$ we use duality of $L^p$ spaces. ~~and~~ For $f \in L^1 \cap L^p$ $(2<p<\infty)$ we have $\|f\|_p = \sup\left\{\left|\int f \overline{\Psi}\right| : \Psi \in L^{p'}\right\}$ where it suffices, by density, to take the sup over all $\Psi \in C_c^\infty$ with $\|\Psi\|_{p'} \le 1$. Now since $K \in L^2$ and because of our choices of $f$ and $\Psi$, the double integral

$\longrightarrow$ $\langle \Psi, Tf \rangle =$ $\int dx\,dy\, \overline{\Psi(x)}\, K(x-y) f(y)$ converges absolutely

Duality argument as exercise!

$$\overset{\text{Fubini}}{=} \int dy\, f(y) \underbrace{\left( \int dx\, K(x-y) \overline{\Psi(x)} \right)}_{\in L^{p'} \text{ by what we've claimed when } p<2} = \langle T^* \Psi, f \rangle$$

$$\le \|f\|_p \cdot A_{p'} \|\Psi\|_{p'} \quad \Rightarrow \quad \|Tf\|_p \le \sup |\langle \Psi, Tf \rangle| / \|\Psi\|_{p'} \le A_{p'} \|f\|_p .$$

<u>Next</u> The strong-type bound $(2,2)$ follows from Plancherel and the (strong!!) assumption $\|\hat{K}\|_\infty \le B \Rightarrow \|T\|_{2,2} \le B$, so ~~it suffi~~ we're left to prove the ("notorious") weak-type $(1,1)$-bd, i.e.,

$|\{|(Tf)(x)| > \alpha\}| \lesssim \frac{\|f\|_{L^1}}{\alpha}$. To this end, we apply the Calderón-Zygmund decomposition (the version in Corollary 2.13) to $|f|$ at height $\alpha$.

$\Rightarrow$ we decompose $\mathbb{R}^d = F \cup \Omega$, $\Omega = \bigcup_{n=1}^\infty Q_n$, $|f(x)| \le \alpha$ on $F$, $|\Omega| \le \frac{1}{\alpha}\int |f|$; $\frac{1}{|Q_n|}\int_{Q_n} |f| \lesssim \alpha$. Now we decompose $f$ according to the decomposition $\mathbb{R}^d = F \cup \Omega$ into parts where $f$ is bdd and where $f$ oscillates and has mean zero (this decomposition is of course <u>not</u> unique and there are many other useful and heuristically meaningful decompositions)

So let us set $f = g + b$ where

$$g(x) = \begin{cases} f(x) & x \in F \\ \frac{1}{|Q_k|}\int_{Q_k} f & x \in Q_k \end{cases}$$ i.e., $|g(x)| \leq \alpha$ on $F$ and $|g(x)| \lesssim \alpha$ on $\Omega$ $\Rightarrow \|g\|_\infty \lesssim \alpha$

$$\Rightarrow b(x) = \begin{cases} 0 & x \in F \\ f - \ldots & x \in Q_k \end{cases}$$ and $\int_{Q_k} b = \int_{Q_k} (f - g) = 0$, i.e., $b$ oscillates to 0!

$= \sum b_k$, $b_k$ supported on $Q_k$

$$\Rightarrow |\{x : |Tf| > \alpha\}| \leq \underbrace{|\{x : |Tg| > \tfrac{\alpha}{2}\}|}_{\leq \frac{\|Tg\|_2^2}{\alpha^2}} + \underbrace{|\{x : |Tb| > \alpha/2\}|}_{\leq |\{x \in F = \Omega^c : |Tb| > \frac{\alpha}{4}\}| + \underbrace{|\Omega|}_{\lesssim \alpha^{-1}\|f\|_1}}$$

$$\underset{\substack{\uparrow \\ T \text{ is } L^2\text{-bdd}}}{\lesssim} \frac{\int |g|^2}{\alpha^2} \leq \frac{1}{\alpha}\int |g| \leq \frac{\|f\|_{L^1}}{\alpha}$$ i.e., $\|Tg\|_{L^{1,\infty}} \lesssim \|f\|_{L^1}$ ok!

Now we only need to control $\|Tb\|_{L^{1,\infty}(F)}$

$$Tb = \sum_k Tb_k = \sum_k \int_{Q_k} K(x-y) b_k(y)\,dy$$

$$= \sum_k \int_{Q_k} \underbrace{(K(x-y) - K(x-y_k))}\, b_k(y)\,dy$$ where $y_k$ is the center of cube $Q_k$.

mean value thm $\Rightarrow$ $|\ldots| \leq |\nabla K(x - \bar{y}_k)| \cdot |y - y_k|$ $\quad$ $\bar{y}_k$ ... variable point on the straight line segment connecting $y (\in Q_k)$ with $y_k$

$$\lesssim \frac{\operatorname{diam} Q_k}{|x - \bar{y}_k|^{d+1}}$$

But now we can appeal to the important property of the CZ decomp (when it was proved with the Whitney's help) that $\operatorname{diam} Q_k \sim \operatorname{dist}(Q_k, F)$

$\Rightarrow$ for fixed $x \in F$, the set of distances $\{|x-y|\}$ as $y$ ranges over $Q_k$ are all comparable with each other

$$\Rightarrow |Tb_k(x)| \lesssim \operatorname{diam} Q_k \int_{Q_k} \frac{|b_k(y)|}{|x-y|^{d+1}}\,dy$$ for $x \in F$, and $|x-y| \sim \operatorname{dist}(F, Q_k)$

$$\delta(y) = \operatorname{dist}(y, F)$$

$$\underset{\int |b_k| \leq \alpha |Q_k|}{\lesssim} \frac{\alpha \cdot \operatorname{diam} Q_k \cdot |Q_k|}{\operatorname{dist}(Q_k, F)^{d+1}} \lesssim \alpha \int_{Q_k} \frac{\delta(y)}{|x-y|^{d+1}}\,dy$$

Thus, for $x \in F$, we majorized $|(Tb_k)(x)|$ by the Marcinkiewicz integral of the beginning of this chapter (Thm 3.1) + Lemma 3.2

$$\Rightarrow |\{x \in F : |Tb| > \tfrac{\alpha}{4}\}| \leq \frac{1}{\alpha}\int_F dx\, |Tb| \lesssim \int_F dx \int_\Omega dy \frac{\delta(y)}{|x-y|^{d+1}} \underset{\text{Lemma 3.2}}{\lesssim} |\Omega| \lesssim \frac{\|f\|_1}{\alpha}$$ $\square$

Remarks • The meaning of oscillation can be adapted to the operator in question more directly, by splitting the $b_i$ once more, e.g., in

leads to a generalized Hörmander condition, see below $\rightarrow$ $b_i = (1-\psi_{h_i}) b_i + \psi_{h_i} b_i$ where, e.g., $\psi_{h_i} = e^{-2^{\alpha h_i}|p|^\alpha}$, $h_i = \log_2 \frac{\operatorname{diam} Q_i}{\sqrt{d}}$

($\psi_{h_i} \sim$ approximation of identity, instead of $\delta$) for $(-\Delta)^{\alpha/2}$

$\rightarrow$ $\mathcal{F}(|p|^\alpha) \sum_i \psi_{h_i} b_i$ can usually be controlled well in $L^2$ norm and

one has to ~~use~~ exploit cancellations in $\|\mathcal{F}(|p|^\alpha) \sum_i (1-\psi_{h_i}) b_i\|_{L^{1,\infty}} \lesssim \|f\|_1$

• The above theorem is somewhat unsatisfactory yet for two reasons:

a) we <u>assumed</u> that $T$ is $L^2$ bdd (although we could have used any other $p \neq 2$, too). This is a very strong assumption; we will remedy this defect in a moment and will later (in a chapter on almost orthogonality) systematically study conditions on $L^2$-bddness of singular integral operators (of CZ-type)

b) To obtain the weak $L^1$-bddness we assumed the <u>cancellation condition</u> $|\nabla K| \lesssim |x|^{-d-1}$, which somewhat reminds one of $C^1$-conditions. We therefore expect to weaken this and replace it by a Lipschitz or Hölder condition like $\int |K(x,y) - K(x,z)| dx \lesssim |y-z|^\alpha$

In fact, we shall only need <u>Hörmander's condition</u>, see below

$$\cancel{\int |K(x-y) - K(x)| dx} \quad \int_{|x-z| \geq 2|y-z|} |K(x-y) - K(x-z)| dx \leq 1 \quad \text{uniformly in } y, z \text{ in } \mathbb{R}^d$$

(which is of course satisfied by the mean value thm whenever $|\nabla K| \lesssim |x|^{-d-1}$)

<u>Corollary 3.8</u> The results of Thm 3.7 hold with

$$\int_{|x-z|>2|y-z|} |K(x-y) - K(x-z)| \leq B' \quad \text{instead of } |\nabla K(x)| \leq \frac{B}{|x|^{d+1}}$$

(* <u>Calderón-Zygmund kernel</u>: replace Hörmander's condition by the Hölder-typ estimate $|K(x-y) - K(y)| \lesssim \frac{|y|^\delta}{|x-y|^{d+\delta}}$ $\forall\, |x| > 2|y|$

<u>CZ operator</u>: linear, $L^2$-bdd operator s.t. $\exists$ singular integral kernel satisfying $|K(x)| \lesssim |x|^{-d}$ and $|K(x-y) - K(x)| \lesssim \frac{|y|^\delta}{|x-y|^{d+\delta}}$ )

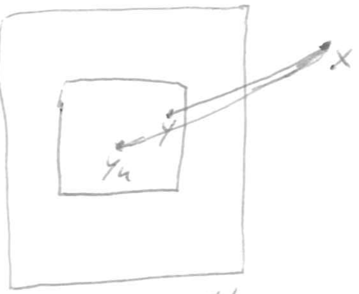
Proof We merely need to re-verify that $T$ has weak-type $(1,1)$.

This time, we won't use that $\operatorname{diam} Q_k \sim \operatorname{dist}(Q_k, F)$ (if only because we want to show that this fact is not really necessary here!)

(*) $\rightarrow$ instead, we'll consider $2\sqrt{d}$-dilates $Q_k^*$ of the $Q_k$

Clearly (i) $Q_k \subseteq Q_k^*$, $|\Omega^*| \leq (2\sqrt{d})^d |\Omega|$

(ii) If $x \notin Q_k^* \Rightarrow |x-y_k| > 2|y-y_k|$ for all $y \in Q_k$




( $\sqrt{a^2+b^2} \overset{a=b}{=} \sqrt{2}|a|$ ; $\sqrt{\sum a_j^2} \underset{a_j=a}{=} \sqrt{d}\,|a|$ )

$\max_{y \in Q_k} |y-y_k| = \sqrt{d} \cdot \underset{\text{length}}{\ell_k}$

(*) another difference is that we'll not majorize $Tb$ by Marcinkiewicz' distance integral, but estimate it directly $\rightarrow$ as a consequence, a favourable estimate is obtained on $F^* = (\Omega^*)^c$ instead of $F$.

Now concretely As in the proof of the theorem, $Tb_k(x) = \int_{Q_k} \left(K(x-y) - K(x-y_k)\right) b_k(y)\,dy$

and $\int_{F^*} |Tb(x)|\,dx \leq \sum_k \int_{(Q_k^*)^c} dx \int_{Q_k} dy\, |K(x-y) - K(x-y_k)|\,|b(y)|$

$\hookrightarrow |x-y_k| > 2|y-y_k|$

$\overset{\text{Hörmander's cond}}{\leq} B' \sum_k \int_{Q_k} dy\, |b(y)| \leq C B' \|f\|_1$ which brings us back to the proof of Theorem 3.7 ∎

Now let's get rid of the a-priori $L^2$-bddness in order to treat p.v.-type integrals (recall Hilbert transform) which exist because of the cancellation of positive and negative values

However, from what we've done ~~so~~ so far, it's now a relatively simple matter to obtain the following

Theorem 3.9 Suppose that the kernel $K(x-y)$ satisfies the conditions

$$|K(x)| \le B|x|^{-d} \quad \text{and} \quad \int_{R_1\le|x|\le R_2} K(x)\,dx = 0 \quad 0<R_1<R_2<\infty$$

(cancellation cond)

$$\int_{|x-z|>2|y-z|} |K(x-y)-K(x-z)|\,dx \le B$$

For $f\in L^p(\mathbb{R}^d)$, $1<p<\infty$ let ~~$T_\epsilon f)(x) := \int_{|x-y|>}$~~

$$(T_\epsilon f)(x) := \int_{|x-y|>\epsilon} K(x-y)f(y)\,dy$$

Then $\|T_\epsilon f\|_p \le A_p \|f\|_p$ and $T_\epsilon f \xrightarrow[\epsilon\to 0]{\|\cdot\|_p} Tf$ and $\|Tf\|_p \le A_p\|f\|_p$

(by FAP I)
uniform bddness

Using the new ~~condit~~ cancellation ~~condition~~ condition, the Hörmander cond and $|K(x)|\le |x|^{-d}$, we'll prove the $L^2$-bddness which, in turn (using the previous assertions) yields the $L^p$-bddness of the truncated $T_\epsilon$.

The proof of the $L^2$-bddness relies on the following

Lemma 3.10 Suppose $K$ satisfies the conditions of Thm 3.9 and let

$$K_\epsilon(x) = \begin{cases} K(x) & |x|>\epsilon \\ 0 & |x|<\epsilon \end{cases}$$

. Then obviously $K_\epsilon \in L^2(\mathbb{R}^d)$ (for all $\epsilon>0$) and for the Fourier transform we have the estimates

$$\sup_\xi |\widehat{K_\epsilon}(\xi)| \le C_d B,\ \epsilon>0$$

Lemma 3.10 is first proved for $\epsilon=1$. A dilation argument will yield the assertion for all $\epsilon>0$. This indeed shows, that $C_d$ does not depend on $\epsilon$.

Proof First, let $\epsilon = l_\alpha$ and observe that $K_1$ still satisfies the bounds in the hypotheses of Thm 3.9 except that the bounds $B$ may be replaced by $C_\alpha \cdot B$.

Next, $\hat{K}_1(\xi) = \lim_{R\to\infty} \int_{|x|<R} e^{2\pi i x\cdot\xi} K_1(x)\, dx = \underbrace{\int_{|x|<\frac{1}{|\xi|}} dx\, K_1(x) e^{2\pi i x\cdot\xi}}_{=: I_1} + \underbrace{\lim_{R\to\infty} \int_{\frac{1}{|\xi|}<|x|<R} dx\, K_1(x) e^{2\pi i x\cdot\xi}}_{=: I_2}$

By the cancellation condition, $\int_{|x|<1/|\xi|} e^{2\pi i x\cdot\xi} K_1(x) dx = \int_{|x|<|\xi|^{-1}} (e^{2\pi i x\cdot\xi} - 1) K_1(x) dx$

$$\underset{\text{mean value thm}}{\overset{\text{''}}{\lesssim}} |\xi| \int_{|x|<|\xi|^{-1}} |x| \underbrace{|K_1(x)|}_{\lesssim |x|^{-d}} \le C' \cdot B$$

which shows $|I_1| \le C'B$. We're left with $I_2$.

Let $z = z(\xi)$ be such that $e^{2\pi i x\cdot\xi}\big|_{x=z(\xi)} = -1$ which is realized for, e.g.,

$z = \frac{1}{2}\frac{\xi}{|\xi|^2}$ (with ~~$|\xi| = \frac{1}{2|\xi|}$~~ $|z| = \frac{1}{2|\xi|}$) $\quad \left(2\pi i \cdot \frac{1}{2}\frac{\xi\cdot\xi}{|\xi|^2} = i\pi \checkmark\right)$

$\Rightarrow \int dx\, e^{2\pi i x\cdot\xi} K_1(x) = \frac{1}{2}\int dx\, e^{2\pi i x\cdot\xi}(K_1(x) - K_1(x-z))$ for this $z = z(\xi)$

$$\Rightarrow \lim_{R\to\infty} \int_{\frac{1}{|\xi|}<|x|<R} dx\, e^{2\pi i x\cdot\xi} K_1(x) = \frac{1}{2}\lim_{R\to\infty} \int_{\frac{1}{|\xi|}<|x|<R} dx\, e^{2\pi i x\cdot\xi}(K_1(x) - K_1(x-z)) \qquad (*)$$

$$- \frac{1}{2} \int_{\substack{\frac{1}{|\xi|}<|x+z| \\ |x|<|\xi|^{-1}}} dx\, e^{2\pi i x\cdot\xi} K_1(x)$$

$\overset{x+z-z}{\overset{\text{''}}{|x|}} > \frac{1}{|\xi|} - \frac{1}{2|\xi|} = \frac{1}{2|\xi|}$

The region $\{x : \frac{1}{|\xi|} < |x+z|, |x| < |\xi|^{-1}\}$ is contained in the spherical shell $\frac{1}{2}|\xi|^{-1} < |x| < |\xi|^{-1}$. ~~Moreover~~ So $\int_{\substack{|\xi|^{-1}<|x+z| \\ |x|<|\xi|^{-1}}} dx\, e^{2\pi i x\cdot\xi} K_1(x)$ is bdd since $|K_1(x)| \lesssim |x|^{-d}$.
$\left(\overset{\text{''}}{\lesssim} |\log |\xi||\right)$

The first integral on RHS of $(*)$ is bdd by $\int_{|x|>|\xi|^{-1}} dx\, |K(x) - K(x-z)| \le CB$ by Hörmander's condition since $|\xi|^{-1} = 2|z|$ $(|x| > 2|z|)$

$\Rightarrow \|\hat{K}_1\|_\infty \lesssim 1$

So far we proved our lemma for $K_1$. To pass to the case of general $K_\epsilon$, we use a simple observation whose significance carries over to the whole theory developed in these notes.

Let $\tau_\epsilon$ be dilation by the factor $\epsilon > 0$, i.e., $(\tau_\epsilon f)(x) = f(\epsilon x)$

$\Rightarrow$ For a convolution operator $T$ ($Tf = \varphi * f = \int \varphi(x-y) f(y)\,dy$), ~~we define~~ the dilated operator is given by $\tau_\epsilon^{-1} T \tau_\epsilon$ which is convolution with $\varphi_\epsilon(x) = \epsilon^{-d} \varphi(x/\epsilon)$

In our case $\tau_{\epsilon^{-1}} T \tau_\epsilon$ has convolution (integral) kernel $\epsilon^{-d} K(x/\epsilon)$, so if $K$ satisfies the assumptions of Thm 3.10, so does $\epsilon^{-d} K(x/\epsilon)$ <u>with the same bounds</u> (by scaling) $|K(x)| \lesssim |x|^{-d} \Rightarrow |K_\epsilon(x)| \lesssim \epsilon^{-d} \frac{\epsilon^d}{|x|^d} = \frac{1}{|x|^d}$

$$\sup_{y,z} \int_{\frac{|x-z|}{\epsilon} > \frac{2|y-z|}{\epsilon}} |K_\epsilon(x-y) - K_\epsilon(x-z)|\,dx = \sup_{y,z} \int_{|x-\frac{z}{\epsilon}| > \frac{2|y-z|}{\epsilon}} |K(x-\tfrac{y}{\epsilon}) - K(x-\tfrac{z}{\epsilon})|$$

$$\overset{\text{replace } (y,z) \text{ by } \epsilon(y,z)}{=} \sup_{y,z} \int_{|x-z| > 2|y-z|} |K(x-y) - K(x-z)|$$

(A similar remark holds for ~~all~~ the assumptions of all the theorems in these notes.)

So let $K' = \epsilon^d K(\epsilon x)$, then $K'$ satisfies the conditions of our lemma with the same bound $B$ and so, if $K_1'(x) = \begin{cases} K'(x) & |x| > 1 \\ 0 & |x| < 1 \end{cases}$ then we know from what proved above that $\|\widehat{K_1'}\|_\infty \le CB$. The FT of $\epsilon^{-d} K_1'(x/\epsilon)$ is $\widehat{K_1'}(\epsilon\xi)$ and so this is again bounded by $CB$; however $\epsilon^{-d} K_1'(x/\epsilon) = K_\epsilon(x)$ so the lemma is completely proved. $\square$

<u>Proof of Theorem 3.9</u>

Since $K$ satisfies the bounds $|K(x)| \lesssim |x|^{-d}$, Hörmander's cond. $\int_{|x-y| > 2|y-z|} |K(x-y) - K(x-z)| \le 1$ and the cancellation cond., so does $K_\epsilon$, at least modulo $d$-dependent constants, as we saw in the proof of Lemma 3.10 (first for $K_1$, then by scaling for all $K_\epsilon$) $\Rightarrow$ by Corollary 3.8, we have $\|T_\epsilon f\|_p \le A_p \|f\|_p$, <u>uniformly</u> in $\epsilon > 0$.

Now, to prove $T_\epsilon f \xrightarrow[\epsilon \to 0]{\|\cdot\|_p} Tf$, we invoke FAP I (uniform bddness principle); by $\|T_\epsilon f\|_p \le \|f\|_p$, it suffices to prove the $L^p$-conv. on a dense subset of $L^p$, e.g. $C_c^1$.

So let $f_1 \in C_c^1$, then

$$(T_\epsilon f_1)(x) = \int_{|y|>\epsilon} K(y) f_1(x-y)\,dy = \int_{|y|\geq 1} K(y) f_1(x-y)\,dy + \int_{1>|y|>\epsilon} K(y) f_1(x-y)\,dy$$

$$\overset{\text{cancellation cond}}{=} \underbrace{\int_{|y|\geq 1} K(y) f_1(x-y)\,dy}_{\in L^p \text{ by Young}} + \underbrace{\int_{1>|y|>\epsilon} K(y) \underbrace{[f_1(x-y) - f_1(x)]}_{\overset{\text{MVT}}{\leq} \|\nabla f_1\|_\infty |y|}\,dy}$$

$1 + \frac{1}{p} = 1 + \frac{1}{p}$ $\quad \|\cdot\|_p \leq \|K\|_p \|f_1\|_1$

$$\xrightarrow[\epsilon \to 0]{\text{uniformly in } x} \int_{|y|<1} K(y) f_1(x-y)\,dy \quad \forall x \quad \text{(dominated convergence)}$$

$\Rightarrow T_\epsilon f_1 \xrightarrow[\epsilon\to 0]{\|\cdot\|_p} T f_1$. So again, use density and write $f = \underset{C_c^1}{f_1} + f_2$ with $\|f_2\|_p < \delta$.

If we apply $\underbrace{\|T_\epsilon f_2\|_p \leq A_p \|f_2\|_p}_{\leq A_p \delta \to 0}$, it follows that $\lim_{\epsilon\to 0} T_\epsilon f$ exists in $L^p$ norm and the limiting operator $T$ obviously satisfies $\|Tf\|_p \leq A_p \|f\|_p$ ∎

We have thereby established the $L^p$ boundedness of the Hilbert transform whose kernel is $\frac{1}{\pi x}$, $x \in \mathbb{R}^1$ $\left( \int_{-\epsilon}^{\epsilon} \frac{1}{x} = 0, \quad \int_{|x|>2|z|} dx \left(\frac{1}{x} - \frac{1}{x-z}\right) \underset{\text{e.g.}}{\overset{z>0}{=}} -\int_{|x|>2z} dx \frac{z}{x^2 - xz} \overset{x \to xz}{=} -\int_{|x|>2} \frac{dx}{x^2 - x} \overset{\text{!!}}{\lesssim} 1 \right)$

$\Rightarrow \lim_{\epsilon\to 0} \frac{1}{\pi} \int_{|y|>\epsilon} \frac{f(x-y)}{y}\,dy$ exists in $L^p$ norm for all $1 < p < \infty$ (but not in $L^1, L^\infty$ as the example $\mathbb{1}_{[a,b]}(x)$ showed)

## 3.5 Singular integral operators which commute with dilations



Besides the group of translations, there's the group of dilations that commute with certain singular integrals. (Sometimes there's also modulation symmetry as in the problem of ptwise convergence of Fourier series.)

We'll discuss in the following operators that satisfy $\underbrace{\tau_{\epsilon^{-1}} T \tau_\epsilon}_{\text{kernel } \epsilon^{-d} K(\frac{x}{\epsilon})} = \underbrace{T}_{\text{kernel } K(x)}$

i.e., $K$ is homogeneous of degree $-d$, or, put differently,

$$K(x) = \frac{\Omega(x)}{|x|^n} \quad \text{where } \Omega \text{ is homogeneous of degree } 0, \text{ i.e., } \Omega(\epsilon x) = \Omega(x) \quad \left(\text{such as } \Omega(x) = \frac{x_j}{|x|}\right)$$

This condition on $\Omega$ is equivalent with the fact that it's constant on rays emanating from the origin; in particular, $\Omega$ is completely determined by its restriction on $S^{d-1}$.

Let's try to reinterpret the conditions of Thm 3.9 in terms of $\Omega$.

$$\left.\begin{array}{l} |K(x)| \lesssim |x|^{-n} \\ \displaystyle\int_{|x-y|>2|y-z|} |K(x-y) - K(x-z)| \lesssim 1 \end{array}\right\} \Rightarrow \|\Omega\|_\infty \lesssim 1 \text{ and } \int_{S^{d-1}} \Omega \, d\omega \lesssim 1$$

$$\int_{R_1 < |x| < R_2} K(x) dx = 0 \iff \int_{S^{d-1}} \Omega(x) \, d\omega = 0.$$

The (precise) Hörmander condition is not easily restated in terms of $\Omega$. It is, however obvious that it requires a certain (weak) continuity on $\Omega$. Here, we shall content ourselves in treating the case where $\Omega$ satisfies the following "Dini-type" condition:

$$\text{If } \sup_{\substack{|x-x'| \le \delta \\ |x|=|x'|=1}} |\Omega(x) - \Omega(x')| = \omega(\delta), \text{ then } \int_0^1 \frac{\omega(\delta)}{\delta} d\delta < \infty$$

Clearly, any $\Omega$ which is $C^\alpha$ with $\alpha > 0$ satisfies this.

Theorem 3.11

Let $\Omega$ be homogeneous of degree $0$ s.t. $\int_{S^{d-1}} \Omega = 0$ " and the Dini condition " $\sup_{\substack{|x-x'|\le\delta \\ |x|=|x'|=1}} |\Omega(x)-\Omega(x')| =: w(\delta) \Rightarrow \int_0 \frac{w(\delta)}{\delta}\, d\delta < \infty$ "

are satisfied. For $1<p<\infty$ and $f\in L^p(\mathbb{R}^d)$ let $(T_\epsilon f)(x) = \int_{|x-y|>\epsilon} \frac{\Omega(x-y)}{|x-y|^d} f(y)\, dy$.

Then: a) $\|T_\epsilon f\|_p \le A_p \|f\|_p$, uniformly in $\epsilon$ and $f$

b) $\lim_{\epsilon\to 0} T_\epsilon f = Tf$ exists in $L^p$ norm and $\|Tf\|_p \le A_p \|f\|_p$

c) If $f\in L^2(\mathbb{R}^d)$, then the Fourier transforms of $f$ and $Tf$ are related by $(Tf)^\wedge(\xi) = m(\xi)\hat f(\xi)$ where $m$ is homogeneous of degree $0$ and, in fact $m(\xi) = \int_{S^{d-1}} \left(\frac{\pi i}{2}\, \mathrm{sgn}(\xi\cdot x) + \log\frac{1}{|x\cdot\xi|}\right)\Omega(x)\, d\omega(x)$

for $|\xi|=1$.

Proof

Clearly, a), b) are immediate consequences of Thm 3.9 once one shows that $\Omega$ as in the Thm satisfies Hörmander's condition. $\rightarrow$ exercise (using Dini's condition)

Now, since $T$ is a convolution operator that is $L^2$-bdd we know that its associated Fourier multiplier $\hat{m}$ is a bdd function (by Plancherel). Moreover, since the convolution kernel is dilation invariant ( $\epsilon^{-d} K(x/\epsilon) = \epsilon^{-d} \frac{\Omega(x/\epsilon)}{|x|/\epsilon^d} = K(x)$) it's clear that the Fourier multiplier is homogeneous of order $0$ $\left(m(\xi) = \int \frac{\Omega(x)}{|x|^d} e^{2\pi i x\cdot\xi} dx \underset{x\mapsto \frac{x}{\epsilon}}{=} m(\xi/\epsilon)\right)$

We'll now deduce the explicit formula for $m(\xi)$.

First, we introduce $K_{\epsilon,\eta}(x) = \begin{cases} \frac{\Omega(x)}{|x|^d} & \epsilon<|x|<\eta \\ 0 & \text{otherwise}\end{cases}$ i.e., $K_{\epsilon,\eta}\in L^1$ and, for $f\in L^2$, we have $(K_{\epsilon,\eta} * f)^\wedge(\xi) = \hat K_{\epsilon,\eta}(\xi)\hat f(\xi)$

We claim (i) $\sup_\xi |\hat K_{\epsilon,\eta}(\xi)| \le A$ uniformly in $\epsilon,\eta$

(ii) if $\xi\ne 0$, then $\lim_{\substack{\epsilon\to 0\\ \eta\to\infty}} \hat K_{\epsilon,\eta}(\xi) = m(\xi)$ with $m(\xi)$ as in the assertion

For this purpose let ~~x~~ $x = r\omega$ and $\xi = k\nu$ (spherical coords.)

We first compute the auxiliary integral

$$I_{\epsilon\eta}(\xi,\omega) = \int_\epsilon^\eta \left[e^{2\pi i \xi\cdot x} - \cos(2\pi k r)\right] \frac{dr}{r}, \quad k>0$$

Its imaginary part, $\int_\epsilon^\eta \frac{\sin(2\pi \xi\cdot x)}{r}\, dr$ is uniformly bdd (exercise 5.4) and converges to $\left(\int_0^\infty dt \frac{\sin t}{t}\right) \operatorname{sgn} \omega\cdot\nu = \frac{\pi}{2} \operatorname{sgn} \omega\cdot\nu$

Analogously, its real part, $\int_\epsilon^\eta \frac{\cos(2\pi i \xi\cdot x) - \cos(2\pi k r)}{r}\, dr$ is bdd in absolute value by a constant times $\log\frac{1}{|\omega\cdot\nu|} + 1$ (by integration by parts once more) and converges to $\log\frac{1}{|\omega\cdot\nu|}$ since $\lim_{\substack{\epsilon\to0\\ \eta\to\infty}} \int_\epsilon^\eta \frac{h(\lambda r) - h(\mu r)}{r}\, dr = h(0)\log\frac{\mu}{\lambda}$ for $\mu,\lambda>0$ and $h$ is an appropriate function. Here, $h(r) = \cos 2\pi r$, $\lambda = k|\omega\cdot\nu|$ and ~~$\mu = R$~~ $\mu = k$ (ex 6.1)

$$\left( \int_\epsilon^\eta dr \int_\mu^\lambda dt\, h'(rt) = \underbrace{\int_{\mu\eta}^{\lambda\eta} dt\, t^{-1} h(t)}_{\xrightarrow{\eta\to\infty} 0} - \underbrace{\int_\mu^\lambda dt\, t^{-1} h(\epsilon t)}_{\xrightarrow{\epsilon\to0} -\int_\mu^\lambda dt\, t^{-1} h(0) = \log\frac{\mu}{\lambda} h(0)} \right)$$

Now $\hat K_{\epsilon\eta}(\xi) = \int_{S^{d-1}} d\omega \int_\epsilon^\eta dr\, r^{-1} e^{2\pi i x\cdot\xi}\, \Omega(\omega) = \int_{S^{d-1}} d\omega\, \Omega(\omega)\, I_{\epsilon\eta}(\xi,\omega)$

$\int \Omega(\omega)\, d\omega = 0$

$\to$ can introduce $\cos(2\pi k r)$

Since $|I_{\epsilon,\eta}(\xi,\omega)| \lesssim 1 + |\log|\omega\cdot\nu|| \to |\hat K_{\epsilon\eta}(\xi)| \lesssim \int_{S^{d-1}} d\omega\, \underbrace{|\Omega(\omega)|}_{\lesssim 1}\left(1 + \log\frac{1}{|\omega\cdot\nu|}\right) \lesssim 1$,

thereby proving the uniform bddness, i.e. (i).

But in view of the limit $\lim_{\substack{\epsilon\to0\\ \eta\to\infty}} I_{\epsilon,\eta}(\xi,\omega) = \log\frac{1}{|\omega\cdot\nu|} + i\frac{\pi}{2}\operatorname{sgn}\omega\cdot\nu$ and dominated convergence, we obtain the claim

$$m(\xi) = \lim_{\substack{\epsilon\to0\\ \eta\to\infty}} \hat K_{\epsilon,\eta}(\xi) = \int_{S^{d-1}} d\omega\, \Omega(\omega)\Big(\underbrace{\log\frac{1}{|\omega\cdot\nu|}}_{\text{corresponds to } \Omega_e} + \underbrace{\frac{i\pi}{2}\operatorname{sgn}\omega\cdot\nu}_{\text{corresponds to } \Omega_o}\Big)$$

Finally, by Plancherel, $K_{\epsilon,\eta} * f$ converges in $L^2$ as $\epsilon\to0$, $\eta\to\infty$ whenever $f\in L^2$ and the limit equals $m(\xi)\hat f(\xi)$. However, if we keep $\epsilon$ fixed and let $\eta\to\infty$, then clearly $\int K_{\epsilon,\eta}(x-y) f(y)\, dy$ converges everywhere to $\cancel{\int_{y}} \int_{|x-y|<\epsilon} K(x-y) f(y)\, dy = (T_\epsilon f)(x)$.

So letting $\epsilon\to0$, we obtain ~~[illegible] (a) and (b)~~ c)

which concludes the proof of Thm 3.1 $\boxtimes$

Remark 3.12 (Even and odd kernels)

Proof of Thm 3.11(c) holds under fairly general conditions on $\Omega$. Write $\Omega = \underset{\text{ev}}{\Omega_e} + \underset{\text{odd}}{\Omega_o}$ with $\Omega_e(x) = \Omega_e(-x)$, $\Omega_o(-x) = -\Omega_o(x)$, then by the uniform bddness of $\int_a^b \frac{\sin r}{r}\,dr$, we required only $\int_{S^{d-1}} |\Omega_o(w)|\,dw < \infty$, i.e., $\Omega_o \in L^1(S^{d-1})$.

On the other hand, for the even part, we merely required the uniform boundedness of $\int_{S^{d-1}} |\Omega_e(w)| \log \frac{1}{|w\cdot \nu|}\,dw$ ~~$\forall \nu$~~

## 3.6 A.e. convergence of homogeneous singular integral operators

Thm 3.11 guaranteed the existence of $\lim_{\epsilon\to 0} \int \frac{\Omega(y)}{|y|^{d+1}} f(x-y)\,dy$ in $L^p$ norm.

The natural counterpart of this result is that of convergence pointwise a.e. The following result will be, in effect, a consequence of $L^p$ convergence and $L^p$-bddness (and weak type (1,1) bounds) on the associated maximal operator by means of FAP II.

**Thm 3.12** Suppose $\Omega$ satisfies the assumptions of Thm 3.12 (cancellation cond, Dini cond, homogeneous of degree 0). For $1 \le p < \infty$, consider for $f \in L^p$

$\int_0^1 \frac{w(\delta)}{\delta}\,d\delta < \infty$ whenever $|\Omega(x)-\Omega(x')| \le w(\delta)$, $|x-x'| \le \delta$, $|x| = |x'| = 1$

$$(T_\epsilon f)(x) = \int_{|x-y|>\epsilon} \frac{\Omega(x-y)}{|x-y|^d} f(y)\,dy \quad (\epsilon > 0)$$ (which converges absolutely for all $x$)

Then: a) $\lim_{\epsilon\to 0} (T_\epsilon f)(x)$ exists for almost every $x$

b) Let $(T^* f)(x) = \sup_{\epsilon>0} |(T_\epsilon f)(x)|$. If $f \in L^1(\mathbb{R}^d)$, then $\|T^* f\|_{L^{1,\infty}} \lesssim \|f\|_1$

c) If $1 < p < \infty$, then $\|T^* f\|_p \le A_p \|f\|_p$.

Strategy c) follows from convergence in $L^p$ and certain general properties of approximations of the identity

## Proof of Thm 3.13

c) ($\|T^* f\|_p \lesssim \|f\|_{p}^{?}$, $1<p<\infty$)

By Thm 3.11 (b), we know the existence of $(Tf)(x) = \lim_{\epsilon\to 0} (T_\epsilon f)(x)$ in $L^p$ norm.

The assertion will follow, once we show $(T^* f)(x) \lesssim (M \circ Tf)(x) + (Mf)(x)$

Let $0 \le \varphi \in C_c^\infty(\mathbb{R}^d)$ be a bump fct (radially decreasing) with $\operatorname{supp} \varphi \subseteq B_0(1)$ and $\int_{\mathbb{R}^d} \varphi = 1$. Denote $K_\epsilon(x) := \begin{cases} K(x) = \frac{\Omega(x)}{|x|^d} & \text{if } |x| \ge \epsilon \\ 0 & \text{if } |x| < \epsilon \end{cases}$ and define

only sees singular part at $|x|=1$ and smears it a bit out

$\Phi := \varphi * K - K_1$ where

$$K_\epsilon f = (\varphi_\epsilon * K - \Phi_\epsilon) * f$$

$$\sup_{\epsilon>0} \lesssim M(Tf) + Mf$$

$$\varphi * K = \lim_{\epsilon\to 0} \varphi * K_\epsilon = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} K(x-y)\varphi(y)\,dy$$

Of course, we wish to apply Thm 2.6 ($\sup_{\epsilon>0} |(\varphi_\epsilon * f)(x)| \lesssim (Mf)(x)$) and so we need to show that the least decreasing radial majorant of $\Phi$ belongs to $L^1$.

Observe that for $|x|<1$, we already have ~~$\Phi(x) = (\varphi * K)(x) = \int K(y)(\varphi(x-y) - \varphi(x))\,dy$~~

$$\Phi(x) = \varphi * K(x) = \int_{\mathbb{R}^d} K(y)\varphi(x-y)\,dy = \int K(y)\underbrace{(\varphi(x-y) - \varphi(x))}_{\overset{\text{TL}}{\lesssim} \|\nabla\varphi\|_\infty |y|}\,dy \overset{(*)}{\lesssim} 1$$

(large $|y|$ are not dangerous for fixed $x$, as $\varphi$ is compactly supported)

~~$\varphi$ radial, $\mathbb{R}$~~

$$\int_{\mathbb{R}^d} dy\, K(y) = \int_0^\infty dr\, r^{d-1-d} \int_{S^{d-1}} \Omega\, d\omega = 0$$

For $1<|x|<2$, we have $\Phi(x) = \underset{\overset{(*)}{\lesssim} 1}{\varphi * K} - K(x)$ which is bounded for the same reason

bdd for $1<|x|<2$

Finally, for $|x|>2$, we have

$$\Phi = \int K(x-y)\varphi(y)\,dy - K(x) = \int_{\substack{|y|<1 \\ |x|>2|y|}} \overbrace{(K(x-y) - K(x))}^{= \frac{\Omega(x-y)-\Omega(x)}{|x-y|^d} + \Omega(x)\left(\frac{1}{|x-y|^d} - \frac{1}{|x|^d}\right)} \varphi(y)\,dy$$

$\rightsquigarrow$ Hörmander's cond

$$\omega(\delta) = \sup_{\substack{|x-x'|<\delta \\ |x|=|x'|=1}} |\Omega(x) - \Omega(x')|$$

If $|x|>2|y| \Rightarrow \left|\frac{x-y}{|x-y|} - \frac{x}{|x|}\right| \lesssim \frac{|y|}{|x|}$

$\Rightarrow |\Omega(x-y) - \Omega(x)| \le \omega(|y|/|x|)$

$|x-y| > |x| - |y| > |x|/2$

$$\frac{1}{|x-y|^d} - \frac{1}{|x|^d} \le |y| \sup_{\xi \in [x-y,x]} |\xi|^{-d-1} \lesssim \frac{|y|}{|x|^{d+1}}$$

$$\overset{(*)}{\lesssim} \int_{|y|<1} \left[\omega\left(c\frac{|y|}{|x|}\right)\frac{dy}{|x|^d} + \|\Omega\|_\infty \underbrace{\left|\frac{1}{|x-y|^d} - \frac{1}{|x|^d}\right|}_{\lesssim |x|^{-d-1}\cdot|y| \le |x|^{-d-1}}\right] dy$$

$$\overset{\omega \text{ increasing fct}}{\lesssim} \omega\left(\frac{c}{|x|}\right)\cdot\frac{1}{|x|^d} + |x|^{-d-1} \Leftarrow \text{radially decreasing and integrable}$$

Since $\int_0^1 \frac{\omega(\delta)}{\delta}\,d\delta < \infty$ this shows that

$$\int_{|x|>2} \omega\left(\frac{c}{|x|}\right)\frac{dx}{|x|^d} \le \int_2^\infty \omega\left(\frac{c}{r}\right)\frac{dr}{r} = \int_0^1 \omega(\delta)\frac{d\delta}{\delta} < \infty$$

~~Since the~~ Now, dilation invariance + some technicalities

Since the integral operator $\varphi \mapsto \varphi * K$ commutes with dilations, we have

$$\Phi_\epsilon = \varphi_\epsilon * K - K_\epsilon \quad \text{where } \Phi_\epsilon(x) = \epsilon^{-d}\Phi(x/\epsilon).$$

Now we claim that for any $f \in L^p(\mathbb{R}^d)$, $1<p<\infty$, we have

$$(\varphi_\epsilon * K) * f = (Tf) * \varphi_\epsilon \quad \text{for all } x \in \mathbb{R}^d. \quad (*)$$

In fact, notice that $((\varphi_\epsilon * K_\delta) * f)(x) = (T_\delta f * \varphi_\epsilon)(x)$, $\delta > 0$ $(**)$

because both sides of $(**)$ ~~$\varphi_\epsilon * K_\delta - K_\epsilon = \Phi$~~ are equal for each $x$ to the absolutely convergent double integral $\int_{\mathbb{R}^d} dz \int_{|y|>\delta} dy\, K(y) f(z-y) \varphi_\epsilon(x-z)$.

Since also $\varphi_\epsilon \in L^{p'}$ with $1<p'<\infty$ we have $\varphi_\epsilon * K_\delta \xrightarrow{L^{p'}} \varphi_\epsilon * K$ (since $T_\delta f \xrightarrow{L^p} Tf$)

~~thereby~~ thereby showing $(*)$. Thus, by def. of $\Phi_\epsilon = \varphi_\epsilon * K - K_\epsilon$, we have

$$T_\epsilon f = Tf * \varphi_\epsilon - \Phi_\epsilon * f \overset{\sup_\epsilon}{\lesssim} M(Tf) + Mf$$

Passing to the supremum over all $\epsilon > 0$ and applying Thm 2.6 a) yields the desired majorization of $T^*$ in terms of the Hardy-Littlewood maximal operator. ~~a~~ Thus, the $L^p$-bddness of $T$ and $M$ establish c).

b) proof similar to the one of ~~Thm 3.~~ Corollary 3.8 of Thm 3.7 (p. 11)

For given $\alpha > 0$ split $f = g + b$ with $g(x) = \begin{cases} f(x) & x \in F \\ \frac{1}{|Q_k|}\int_{Q_k} f & x \in Q_k \end{cases}$

and let $Q_k^*$ the $2\sqrt{d}$ dilate of $Q_k$.

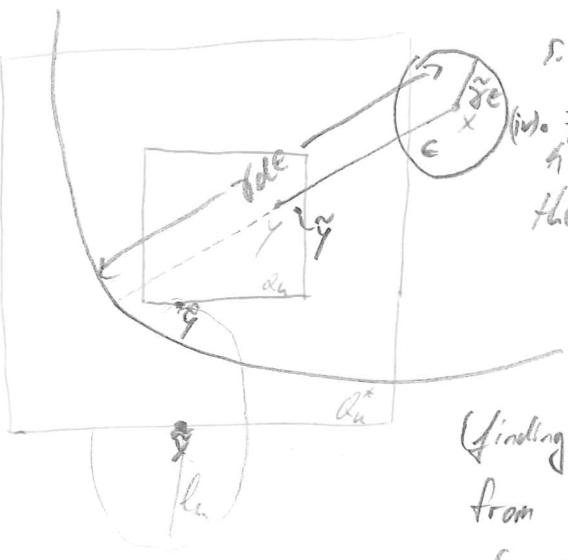
Let's make some geometrical remarks concerning the $Q_k^*$ (we recall (i)+(ii) from p. 12)

(i) $Q_k \subset Q_k^*$, $|\Omega^*| \leq (2\sqrt{d})^d |\Omega|$

(ii) If $x \notin Q_k^* \Rightarrow |x - y_k| > 2|y - y_k|$ for all $y \in Q_k$

(iii) Suppose $x \in (Q_k^*)^c$ (in particular, that's the case when $x \in F^* = (\Omega^*)^c$) and assume that for some $y \in Q_k$, $|x-y| = \epsilon$. Then the closed ball $\overline{B_x(\gamma_d \epsilon)}$ contains $Q_k$ (for some $\gamma_d$ indep of $Q_k$), i.e., $Q_j \subset \overline{B_x(\gamma_d \epsilon)}$

(iv) Under the hypothesis of (iii), we have $|x-y| \geq \gamma_d' \epsilon$ for every $y \in Q_k$




(iii). Given $\epsilon$, find $y$ s.t. $|x-y| = \epsilon \to$ find $\gamma_d$ s.t. $B_x(\gamma_d \epsilon) \supseteq Q_k$

(iv). For any $\tilde{y} \in Q_k$ $\exists\, \tilde{\gamma}_d$ s.t. $\tilde{y} \in B_x(\tilde{\gamma}_d \epsilon)^c$
the above given $\epsilon$

i.e. $|x-\tilde{y}| > \tilde{\gamma}_d \epsilon$

(finding $\gamma_d$ becomes easier the farther $x$ is away from $Q_k^*$! "extreme" case: $x$ lies on boundary of $Q_k^*$ and $\epsilon$ is smallest dist between $Q_k$ and $Q_k^\circ$, i.e., $\epsilon \sim l_k$)

Now, with these observations and ~~the~~ following the arguments on ~~[illegible]~~ pp. 10–12 (on proof of weak-type estimate for $T$), it suffices to prove

$$(*) \quad \sup_{\epsilon>0} |(T_\epsilon b)(x)| \lesssim \underbrace{\sum_k \int_{Q_k} |K(x-y) - K(x-y_k)|\,|b(y)|\,dy}_{I} + c_d \underbrace{\sup_{r>0} \frac{1}{|B_x(r)|} \int_{B_x(r)} |b(y)|\,dy}_{II = Mb}$$

(with $K = \frac{\Omega}{|x|^d}$) for $x \in F^*$

new element of the proof

(Assuming $(*)$, we're done since
($L^2$-bddness for $\|Tg\|_2$ already contained in c); $|(F^*)^c| \lesssim \frac{\|f\|_1}{\alpha}$ ✓)
and
$\| II \|_{L^{1,\infty}} \lesssim \|b\|_{L^1}$ by maximal inequality
$\| I \|_{L^{1,\infty}} \lesssim \|f\|_1$ shown on p. 12

just need to check $|\{ \overset{F^*}{\downarrow} x : \sup_{\epsilon>0} |(T_\epsilon b)(x)| > \alpha \}| \le \overbrace{\frac{1}{\alpha} \int_{F^*} dx \sup_{\epsilon>0} \sum_k \int_{Q_k} |K(x-y) - K(x-y_k)|\,|b(y)|\,dy}^{\text{treated on p. 12}}$
to obtain weak-type ineq.

$+ \underbrace{\|Mb\|_{L^{1,\infty}}}_{\lesssim \|b\|_1 \lesssim \|f\|_1\ ✓}$

So let's prove $(*)$, i.e.,

$$\sup_{\epsilon>0} |(T_\epsilon b)(x)| \le \sum_k \int_{Q_k} |K(x-y)-K(x-y_k)|\,|b(y)|\,dy + c \sup_{r>0} \frac{1}{|B_x(r)|}\int_{B_x(r)} |b(y)|\,dy$$

for all ~~$x \in B$~~ $x \in F^*$

First, fix $x \in F^*$ and $\epsilon > 0$ and observe that the $Q_k$ fall into three classes:

a) For all $y \in Q_k$, we have $|x-y_k| < \epsilon$

b) For all $y \in Q_k$, we have $|x-y_k| > \epsilon$

c) There is a $y \in Q_k$ s.t. $|x-y| = \epsilon$

We now examine $(T_\epsilon b)(x) = \sum_k \int_{Q_k} K_\epsilon(x-y)b(y)\,dy$ in these cases.

Case a) In this case $K_\epsilon(x-y) = 0$ so $\int_{Q_k} K_\epsilon(x-y)b(y)\,dy = 0$

Case b) Here, $K_\epsilon(x-y) = K(x-y)$, so

$$\int_{Q_k} K(x-y)b(y)\,dy = \int_{Q_k} (K(x-y)-K(x-y_k))b(y)\,dy \overset{(c)}{\le} \int_{Q_k} |K(x-y)-K(x-y_k)|\,|b(y)|\,dy$$

which is the first term appearing on RHS of $(*)$.

Case c) $\left|\int_{Q_k} K_\epsilon(x-y)b(y)\,dy\right| \le \int_{Q_k} |K_\epsilon(x-y)|\,|b(y)|\,dy = \int_{B_x(r)\cap Q_k} |K_\epsilon(x-y)|\,|b(y)|\,dy$ by the geom observation (iii) with $r = \gamma_d \epsilon$.

But since $\|\Omega\|_\infty \le 1$ and geom observation (iv), we also have

$$|K_\epsilon(x-y)| \le \frac{|\Omega(x-y)|}{|x-y|^d} \le \frac{B}{(\gamma_d' \cdot \epsilon)^d}$$

and therefore $(\gamma_d' \sim \gamma_d)$

$$\left|\int_{Q_k} K_\epsilon(x-y)b(y)\,dy\right| \lesssim \frac{1}{|B_x(r)|}\int_{B_x(r)} |b(y)|\,dy$$

$\Rightarrow$ Summing over all cubes and taking cases (a)–(c) into account finally gives $(*)$ with $r = \gamma_d \epsilon$, so we're done!

Finally, we prove a), which is, as usual an immediate consequence

$T_\epsilon f(x) \to (Tf)(x)$ ptwise a.e. for $f \in L^p$, $1 \le p < \infty$

of FAP and $L^2$-bddness (resp. weak-type (1,1)) of $T^*_2$ + ptwise conv. on dense fct spaces.

More precisely, for any $f \in L^p$, $1 \le p < \infty$, let

$$(\Lambda f)(x) = \left| \overline{\lim_{\epsilon \to 0}} (T_\epsilon f)(x) - \underline{\lim_{\epsilon\to 0}} (T_\epsilon f)(x) \right|, \text{ then clearly } (\Lambda f)(x) \le 2(T^* f)(x)$$

Now write $f = f_1 + f_2$ with $f_1 \in C^1_c$ and $\|f_2\|_p \le \delta$ for given $\delta > 0$.

We already remarked ~~ex~~ in Thm 2.6 d) and the proof of Thm 3.9 that $T_\epsilon f_1$ converges in fact uniformly as $\epsilon \to 0$, so $(\Lambda f_1)(x) \equiv 0$.

But $(\Lambda f)(x) \le (\Lambda f_1)(x) + (\Lambda f_2)(x)$ so $\|\Lambda f\|_p \le \|\Lambda f_2\|_p \lesssim 2A_p \|f_2\|_p \lesssim \delta$, $1 < p < \infty$

so $\Lambda f_2 = 0$ a.e., so $\Lambda f = 0$ a.e., so $\lim_{\epsilon \to 0} (T_\epsilon f)(x)$ exists a.e. for $1 < p < \infty$.

For $p = 1$, we similarly get ~~$|\{\Lambda f > \alpha$~~ $|\{x : (\Lambda f)(x) > \alpha\}| \lesssim \frac{1}{\alpha}\|f_2\|_1 \le \frac{\delta}{\alpha}$, so again $\Lambda f = 0$ a.e. which implies $\lim_{\epsilon \to 0} (T_\epsilon f)(x)$ exists a.e. $\boxed{}$

## 3.7 Vector-valued analogs

(see also Grafakos Sect. ~~•~~5.6 for more details)

The results of these lectures can be extended to functions taking their values in a general Hilbert space (instead of $\mathbb{R}$ or $\mathbb{C}$). This is good to know when we're going to discuss Littlewood-Paley theory later.

### Short review of certain aspects of integration theory

- $\mathcal{H}$... separable Hilbert space (often $\ell^2$)
- $f : \mathbb{R}^d \to \mathcal{H}$ is called <u>measurable</u> if the scalar-valued functions $\langle f(x), \varphi \rangle_{\mathcal{H}}$ are $\mathbb{C}$-measurable for any vector $\varphi \in \mathcal{H}$
- If $f$ is $\mathcal{H}$-measurable, then ~~so is~~ $\underbrace{|f(x)|_{\mathcal{H}}}_{= \langle f(x), f(x) \rangle_{\mathcal{H}}^{1/2}}$ is $\mathbb{C}$-measurable
- $L^p(\mathbb{R}^d; \mathcal{H})$... equivalence classes of measurable fcts $f : \mathbb{R}^d \to \mathcal{H}$ with the property that the norm $\|f\|_p^p = \int_{\mathbb{R}^d} |f(x)|^p_{\mathcal{H}}\, dx < \infty$ $p < \infty$

$$\|f\|_\infty = \text{ess-sup}\, |f(x)|_{\mathcal{H}} < \infty \quad p = \infty$$

- $B(\mathfrak{h}_1, \mathfrak{h}_2)$ ... Banach space of bounded linear operators from $\mathfrak{h}_1$ to $\mathfrak{h}_2$
- $f : \mathbb{R}^d \to B(\mathfrak{h}_1, \mathfrak{h}_2)$ is called measurable if $f(x)\varphi$ is an $\mathfrak{h}_2$-measurable function for all $\varphi \in \mathfrak{h}_1$. In this case, $\underbrace{|f(x)|_{B(\mathfrak{h}_1, \mathfrak{h}_2)}}_{\text{operator norm}}$ is $\mathbb{C}$-measurable

and we can define the space $L^p(\mathbb{R}^d : B(\mathfrak{h}_1, \mathfrak{h}_2))$ as before

- The usual facts about convolution hold in this setting.

For instance, suppose $K(x) \in L^q(\mathbb{R}^d : B(\mathfrak{h}_1, \mathfrak{h}_2))$ and $f(x) \in L^p(\mathbb{R}^d : \mathfrak{h}_1)$. Then
$g(x) = \int_{\mathbb{R}^d} K(x-y) f(y) dy$ converges in $|\cdot|_{\mathfrak{h}_2}$-norm for a.e. $x$ and
$|g(x)|_{\mathfrak{h}_2} \leq \int |K(x-y)|_{B(\mathfrak{h}_1, \mathfrak{h}_2)} |f(y)|_{\mathfrak{h}_1} dy$ and $\|g\|_r \leq \|K\|_q \|f\|_p$ $\quad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$. $1 \leq r \leq \infty$

- For $f \in L^1(\mathbb{R}^d, \mathfrak{h})$, its Fourier transform is defined as $\hat{f}(\xi) = \int_{\mathbb{R}^d} dx\, e^{-2\pi i x \cdot \xi} f(x)$ and $\hat{f} \in L^\infty(\mathbb{R}^d : \mathfrak{h})$. If $f \in L^1 \cap L^2(\mathbb{R}^d : \mathfrak{h}) \Rightarrow \hat{f} \in L^2(\mathbb{R}^d : \mathfrak{h})$ with $\|\hat{f}\|_2 = \|f\|_2$ and the FT can be extended by continuity to a unitary mapping of $L^2(\mathbb{R}^d : \mathfrak{h})$ to itself. These facts can be obtained from the scalar-valued case by introducing an arbitrary ONB in $\mathfrak{h}$.

Now suppose $\mathfrak{h}_1, \mathfrak{h}_2$ are two given Hilbert spaces, $f(x)$ takes values into $\mathfrak{h}_1$ and $K(x)$ takes values in $B(\mathfrak{h}_1, \mathfrak{h}_2)$. Then $(Tf)(x) = \int_{\mathbb{R}^d} K(y) f(x-y) dy$, whenever defined, takes values in $\mathfrak{h}_2$

<u>Theorem 3.14</u> The results in these notes, in particular Thm 3.7, Cor 3.8, Thm 3.9 and Thm 3.1[illegible], and Thm 3.13, are valid in the more general context where $f$ takes its values in $\mathfrak{h}_1$, $K$ takes its values in $B(\mathfrak{h}_1, \mathfrak{h}_2)$ and $(Tf)$ and $(T_\epsilon f)$ take their values in $\mathfrak{h}_2$ and where throughout the absolute value $|\cdot|$ is replaced by the appropriate Hilbert space norm in $\mathfrak{h}_1$, $B(\mathfrak{h}_1, \mathfrak{h}_2)$ or $\mathfrak{h}_2$, respectively.

This thm is not in any <u>obvious</u> way a corollary of the scalar-valued case; however, its proof consists in nothing but an identical repetition of the arguments given for the scalar-valued case, if we take into account the remarks made ~~in~~ above. (For details, consult, e.g., Grafakos Sect 5.6)

In verifying this seemingly bold assertions, some ~~ele~~ clarifying observations emerge:

a) The final bounds obtained do not depend on the Hilbert spaces $\mathcal{H}_1$, or $\mathcal{H}_2$, but only on $B$ (bounds on the kernel), $p$, and $d$, as in the scalar-valued case

b) Most of the arguments goes through in the even greater generality of Banach space valued functions, appropriately defined ($\to$ Grafakos) The Hilbert space structure is used only ~~to define~~ in the $L^2$ theory when applying the variant of Plancherel's formula described above.

The Hilbert space structure also enters the following ~~corol~~

<u>Corollary 3.15</u> With the assumptions as in Thm 3.14, if in addition $\|Tf\|_2 = c\|f\|_2$, $c>0$, $f \in L^2(\mathbb{R}^d, \mathcal{H}_1)$, then <u>the reverse bounds</u> $\|f\|_p \le A_p' \|Tf\|_p$ for $f \in L^p(\mathbb{R}^d, \mathcal{H}_1)$, $1<p<\infty$, hold.

In particular, this corollary applies to the singular integrals $\frac{\Omega(x)}{|x|^d}$; then the condition required is that the multiplier $m(\xi)$ has constant absolute value. This is the case, e.g., when $T$ is the Hilbert transform, $K(x) = \frac{1}{\pi x}$, $m(\xi) = -i\,\mathrm{sgn}(\xi)$.

Use duality!

<u>Proof</u> First, recall that $L^2(\mathbb{R}^d, \mathcal{H}_j)$ are Hilbert spaces, with associated inner products

$$(f,g)_j = \int \langle f(x), g(x)\rangle_{\mathcal{H}_j}\, dx.$$

Since $T$ is a bdd lin transf from the Hilbert space $L^2(\mathbb{R}^d, \mathcal{H}_1)$ to $L^2(\mathbb{R}^d, \mathcal{H}_2)$, so by the theory of inner products $\exists!$ adjoint transf. $\tilde{T}$ from $L^2(\mathbb{R}^d, \mathcal{H}_2)$ to $L^2(\mathbb{R}^d, \mathcal{H}_1)$ satisfying ~~$\langle Tf_1, f_2\rangle$~~ $(Tf_1, f_2)_2 = (f_1, \tilde{T} f_2)$, $f_j \in L^2(\mathbb{R}^d; \mathcal{H}_j)$

Note that our assumption is equivalent to the identity (by polarization)

$$\underset{\substack{\| \\ (\tilde{T}Tf_1, g)_2}}{(Tf, Tg)_2} = c^2 (f,g), \quad f,g \in L^2(\mathbb{R}^d : \mathcal{H}_1)$$

So the assumption can be restated as $\tilde{T}Tf = c^2 f$, $f \in L^2(\mathbb{R}^d : \mathcal{H}_1)$

$\tilde{T}$ is again an operator of the same kind as $T$ but takes $\mathcal{H}_2$-valued fcts to $\mathcal{H}_1$-valued fcts and its kernel $\tilde{K}(x) = K^*(-x)$ where $^*$ denotes the adjoint of an element in $B(\mathcal{H}_1, \mathcal{H}_2)$

$$\int \overline{g(x)}\, K(x-y) f(y)\, dy = \langle g, Tf \rangle$$
$$= \langle T^* g, f \rangle = \int dy\, f(y) \int dx\, \overline{K(-(y-x))}\, \overline{\overline{g(x)}}$$

for scalar-valued case; for vector-valued case, complex conjugation becomes $^*$ adjoint taking

→ general case by a limiting argument

This being said, we only need to add the remark that $K^*(-x)$ satisfies the same conditions as $K(x)$ and so for it we have the same conclusions as for $K$ with the same bounds. Thus, by $\tilde{T} T f = c^2 f$, we have
$c^2 \|f\|_p = \|\tilde{T} T f\|_p \leq A_p \|Tf\|_p$ proving the assertion with $A_p' = A_p / c^2$ $\square$

## 3.8 Vector-valued inequalities – Theorem of Marcinkiewicz–Zygmund

Certain non-linear expressions appearing in Fourier analysis such as ~~th~~ maximal functions or square functions, can in fact be viewed as linear quantities taking values in a certain Banach space. → systematic study of Banach-valued operators.

Example Let $T : L^p(X,\mu) \to$ measurable fcts on $(Y,\nu)$ and consider the seemingly nonlinear estimate

$$\underbrace{\Big\| \Big( \sum_j |T f_j|^2 \Big)^{1/2} \Big\|_{L^p}}_{\text{square function}} \leq C_p \Big\| \Big( \sum |f_j|^2 \Big)^{1/2} \Big\|_{L^p} \qquad (*)$$

We will now convert this estimate into a linear one by a slight change of view.
Let $L^p(X, \ell^2)$ .. Banach space of all sequences $(f_j)_j$ of measurable fcts on $X$ satisfying $\|(f_j)_j\|_{L^p(X,\ell^2)} = \left( \int_X \left( \sum_j |f_j|^2 \right)^{p/2} d\mu \right)^{1/p} < \infty$ and define a linear operator $T$ acting on such sequences by setting $\vec{T}((f_j)_j) \equiv (T f_j)_{j \in \mathbb{Z}}$

Then $(*)$ is equivalent to the estimate

$$\| \vec{T}((f_j)_j) \|_{L^p(Y,\ell^2)} \leq C_p \| (f_j)_j \|_{L^p(X,\ell^2)}$$

(Sometimes we will write $\vec{f}$ or $\underline{f}$ instead of $(f_j)_{j\in\mathbb{Z}}$ and $\underline{T}$ instead of $\vec{T}$.)

The following result is classical and fundamental in the subject of vector-valued inequalities

**Theorem 3.16** (Marcinkiewicz-Zygmund)

Let $0<p,q<\infty$ and let $(X,\mu)$, $(Y,\nu)$ be two $\sigma$-finite measure spaces. Then:

a) Let $T: L^p(X,\mu)\to L^q(Y,\nu)$ be a bdd linear operator with $\|T\|_{p\to q}=N$.
Then $T$ has an $\ell^2$-valued extension, i.e., for all complex-valued $f_j\in L^p(X)$, we have

$$\Big\|\Big(\sum_j |(Tf_j)(x)|^2\Big)^{1/2}\Big\|_{L^q(Y)} \le C_{pq}\cdot N\, \Big\|\Big(\sum |f_j|^2\Big)^{1/2}\Big\|_{L^p(X)} \qquad (*)$$

for some $C_{pq}$ that satisfies $C_{pq}=1$ when $p\le q$.
Moreover, if $T$ maps real-valued ~~fcts~~ $L^p$-fcts to real-valued $L^q$-fcts with norm $N_{real}$, then $(*)$ holds for real-valued $f_j$ with $N$ replaced by $N_{real}$.

b) Assume $T: L^p(X)\to L^{q,\infty}(Y)$ with norm $M$. Then,

$$\Big\|\Big(\sum_j |Tf_j|^2\Big)^{1/2}\Big\|_{L^{q,\infty}(Y)} \le D_{pq}\, M\, \Big\|\Big(\sum_j |f_j|^2\Big)^{1/2}\Big\|_{L^p(X)} \quad (**) \text{ for some } D_{pq}$$

Moreover, if $T$ maps real-valued $L^p$-fcts to real-valued $L^{q,\infty}$-fcts with norm $M_{real}$, then $(**)$ holds for real-valued $f_j$ with $M$ replaced by $M_{real}$

The proof relies on the following identities.

**Lemma 3.17** For $0<r<\infty$ define the constants $A_r=\left(\frac{\Gamma((r+1)/2)}{\pi^{(r+1)/2}}\right)^{1/r}$, $B_r=\left(\frac{\Gamma(\frac{r}{2}+1)}{\pi^{r/2}}\right)^{1/r}$

Then for $\lambda_1,\dots,\lambda_d\in\mathbb{R}$ we have

$$\Big(\int_{\mathbb{R}^d} |\lambda_1x_1+\dots+\lambda_dx_d|^r e^{-\pi|x|^2}\,dx\Big)^{1/r} = A_r\,(\lambda_1^2+\dots+\lambda_d^2)^{1/2}$$

$dx=dx_1\dots dx_d$

and similarly for all $w_1,\dots,w_d\in\mathbb{C}$, we have

$$\Big(\int_{\mathbb{C}^d} |w_1z_1+\dots+w_dz_d|^r e^{-\pi|z|^2}\,dz\Big)^{1/r} = B_r\,(|w_1|^2+\dots+|w_d|^2)^{1/2}$$

$dz_1\dots dz_d$ and $dz_j=dx_j\,dy_j$ if $z_j=x_j+y_j$.

Proof By homogeneity, we can assume $\lambda_1^2 + .. \lambda_d^2 = 1$   $\hat{e}_1^t = (1, 0, .. 0)$

Let $A \in O(d)$ be an orthogonal $d \times d$ matrix s.t. $A^{-1}\hat{e}_1 = (\lambda_1 .. \lambda_d)^t$

$(\sim \int e^{-\langle x, Ax \rangle} dx)$ $\Rightarrow (Ax)_1 = \cancel{Ax \cdot e_1} = \langle e_1, Ax \rangle = \langle A^t e_1, x \rangle = \lambda_1 x_1 + .. + \lambda_d x_d$

Now change variables $y \equiv Ax$ in the claimed equality and use $|Ax| = |x|$ to obtain

$$\int_{\mathbb{R}^d} |\lambda_1 x_1 + .. + \lambda_d x_d| e^{-\pi |x|^2} dx = \int_{\mathbb{R}^d} |y_1|^r e^{-\pi |y|^2} dy = 2 \int_0^\infty t^r e^{-\pi t^2} dt$$

$$\overset{t \mapsto \sqrt{t}}{=} \int_0^\infty t^{\frac{r-1}{2}} e^{-\pi t} dt = \frac{\Gamma(\frac{r+1}{2})}{\pi^{(r+1)/2}} = A_r^r$$

The proof of the other integral is almost identical; only after change of variables, one computes

$$\int_{\mathbb{C}} |\eta_1|^r e^{-\pi |\eta_1|^2} d\eta_1 \underset{\text{polar coords}}{=} 2\pi \int_0^\infty t^r e^{-\pi t^2} t\,dt \underset{t \mapsto \sqrt{t}}{=} \pi \int_0^\infty t^{r/2} e^{-\pi t} dt = \frac{\Gamma(\frac{r}{2}+1)}{\pi^{\frac{r}{2}+1}} = B_r^r$$

$\square$

Proof of Thm 3.16 We use the second equality of Lemma 3.17.

a) Let $p > q$ and let $B_r$ as in Lemma 3.17. Assume $(f_j)_j$ is indexed by $\mathbb{Z}^+$. We successively use Lemma 3.17, boldness of $T$ and Hölder with exponents $p/q$ and $(p/q)'$ with respect to the measure $e^{-\pi|z|^2} dz$ and Lemma 3.17 again to deduce for $n \in \mathbb{Z}^+$,

$$\| (\sum_{j=1}^n |Tf_j|^2)^{1/2} \|_q^q = (B_q)^{-q} \int_Y d\nu(y) \int_{\mathbb{C}^n} dz |z_1 Tf_1 + ... + z_n Tf_n|^q e^{-\pi|z|^2}$$

$$= (B_q)^{-q} \int_{\mathbb{C}^n} dz \, e^{-\pi|z|^2} \int_Y d\nu(y) \, | T(z_1 f_1 + .. z_n f_n) |^q$$

$$\leq (B_q)^{-q} N^q \int_{\mathbb{C}^n} dz \, e^{-\pi|z|^2} \Big( \int_X d\mu(x) |z_1 f_1 + .. + z_n f_n|^p \Big)^{q/p}$$

$$\overset{\text{Hölder}}{\leq} (B_q)^{-q} N^q \Big( \int_{\mathbb{C}^n} dz \int_X d\mu(x) \, e^{-\pi|z|^2} |z_1 f_1 + .. z_n f_n|^p \Big)^{q/p}$$

$$= (B_q)^{-q} N^q \Big( B_p^p \int_X d\mu(x) \, (\Sigma |f_j(x)|^2)^{p/2} \Big)^{q/p}$$

$$= (B_p / B_q)^q N^q \| (\Sigma |f_j|^2)^{1/2} \|_p^q \; ; \text{ now let } n \to \infty \Rightarrow C_{pq} = \frac{B_p}{B_q}$$

For $p \le q$, we proceed similarly until we reach the expression (we use Minkowski instead of Hölder)

~~$(N/B_q)^q$~~

$$N^q B_q^{-q} \int_{\mathbb{C}^n} dz\, e^{-\pi|z|^2} \left( \int d\mu(x)\, |z_1 f_1 + \dots z_n f_n|^p \right)^{q/p}$$

$$= N^q B_q^{-q} \left\| \int_X d\mu\, |z_1 f_1 + \dots z_n f_n|^p \right\|^{q/p}_{L^{q/p}(\mathbb{C}^n,\, e^{-\pi|z|^2} dz)}$$

$$\overset{\text{Minkowski}}{\le} (N/B_q)^q \left( \int_X \left\| |z_1 f_1 + \dots z_n f_n|^p \right\|_{L^{q/p}(\mathbb{C}^n,\, e^{-\pi|z|^2} dz)} d\mu \right)^{q/p}$$

$$= (N/B_q)^q \left( \int_X d\mu(x) \left( \int_{\mathbb{C}^n} dz\, e^{-\pi|z|^2} |z_1 f_1 + \dots + z_n f_n|^q \right)^{p/q} \right)^{q/p}$$

$$= (N/B_q)^q \left( \int d\mu(x)\, (B_q)^p \left( \sum_{j=1}^n |f_j|^2 \right)^{p/2} \right)^{q/p} = N^q \left\| \left( \sum_{j=1}^n |f_j|^2 \right)^{1/2} \right\|_p^q .$$

now again $n \to \infty$, $C_{pq} = 1$.

If $T$ maps $\mathbb{R}$-valued to $\mathbb{R}$-valued fcts, we instead use the first identity in Lemma 3.17 and replace $B_p$ by $A_p$, $B_q$ by $A_q$ and $N_{\cancel{\text{comp}}}$ by $N_{real}$

b) This will follow from a) and ~~the identity relation~~

$\nu(E)^{\frac{1}{q} - \frac{1}{r} + \frac{1}{r} - \frac{1}{q}} = 1$

$$\|g\|_{L^{q,\infty}} \le \sup_{0 < \nu(E) < \infty} \nu(E)^{\frac{1}{q} - \frac{1}{r}} \left( \int_E |g|^r d\nu \right)^{1/r} \le \left( \frac{q}{q-r} \right)^{1/r} \|g\|_{L^{q,\infty}}, \quad 0 < r < q$$

$E \subseteq Y$ of finite measure

infact, we have equality here (cf Thm 1.2.18) with $r = 1$

Hölder

$\left(\frac{q}{r}\right)' = \left(1 - \frac{r}{q}\right)^{-1} = \frac{q}{q-r}$

$\nu(E)^{(1 - \frac{r}{q})/r} = \nu(E)^{\frac{1}{r} - \frac{1}{q}}$

Using this estimate, we obtain

$$\left\| \left( \sum_j |T f_j|^2 \right)^{1/2} \right\|_{L^{q,\infty}} \le \sup_{0 < \nu(E) < \infty} \nu(E)^{\frac{1}{q} - \frac{1}{r}} \left( \int_E d\nu(y) \left( \sum_j |T f_j|^2 \right)^{r/2} \right)^{1/r}$$

$$= \text{——} \;\text{"}\; \text{——} \left( \int_Y d\nu(y) \left( \sum_j |\mathbb{1}_E T f_j|^2 \right)^{r/2} \right)^{1/r}$$

$$\le \sup_{0 < \nu(E) < \infty} \nu(E)^{\frac{1}{q} - \frac{1}{r}} \|\mathbb{1}_E T\|_{L^p \to L^r}\, C_{pr} \left( \int_X \left( \sum |f_j|^2 \right)^{p/2} d\mu(x) \right)^{\frac{1}{p}} \quad (\square)$$

Since for any $f \in L^p(X)$, we have (by the above estimate)

$$\nu(E)^{1/q - 1/r} \|\mathbb{1}_E T f\|_{L^r} \le \left( \frac{q}{q-r} \right)^{1/r} \|T f\|_{L^{q,\infty}} \le \left( \frac{q}{q-r} \right)^{1/r} M \|f\|_{L^p},$$

it follows that for any $E \subseteq Y$ finitely measurable, we have

$$\nu(E)^{1/q - 1/r} \|\mathbb{1}_E T\|_{L^p \to L^r} \le \left( \frac{q}{q-r} \right)^{1/r} M.$$

$C_{pr} = \begin{cases} B_p / B_r & r < p \\ 1 & r \ge p \end{cases}$

Plugging this in $(\square)$, we obtain the desired conclusion with $D_{pq} = C_{pr} \left( \frac{q}{q-r} \right)^{1/r}$, $0 < r < q$.

If $T$ maps $\mathbb{R}$-valued fcts to $\mathbb{R}$-valued fcts, the arguments of a) apply as well $\square$