Harmonic Analysis (Summer term 2020) ①

1 L^p spaces and interpolation

(L. Grafakos' Classical Fourier Analysis Chapter 1)
(T. Tao's notes on Fourier analysis, lecture 1)
(G. Rey's notes (based on Tao's notes) on Lorentz spaces)

1.1 L^p and weak L^p

The following considerations can be generalized to arbitrary measure spaces X with associated positive, not necessarily finite measures μ on X.

As usual ~~$\|f\|_{L^p(\mathbb{R}^d)} = \int_{\mathbb{R}^d}$~~ $\|f\|_{L^p(X,\mu)} = \left(\int_X |f(x)|^p \, d\mu(x)\right)^{1/p}$, $0<p<\infty$

and $\|f\|_{L^\infty(X,\mu)} = \text{ess-sup}\,|f| = \inf\{B>0 : \mu(\{x\in X : |f(x)|>B\})=0\}$

For $1\le p\le\infty$ we have the triangle ineq $\|f+g\|_p \le \|f\|_p + \|g\|_p$ which is reversed for $0<p<1$ when $f,g\ge 0$. However, the following substitute holds $\|f+g\|_p \le 2^{(1-p)/p}(\|f\|_p+\|g\|_p)$ (*)(exercise) $\Rightarrow$ L^p is a quasi-normed linear space for $0<p<1$.

Moreover, L^p is complete for all $0<p\le\infty$ (Cauchy sequences converge)

$\Rightarrow L^p \begin{cases} \text{Banach} & 1\le p\le\infty \\ \text{quasi-Banach} & 0<p<1 \end{cases}$

Denote by $p' = \frac{p}{p-1}$ (i.e., $\frac{1}{p}+\frac{1}{p'}=1$), $1'=\infty$, $\infty'=1$

Hölder $\|fg\|_1 \le \|f\|_p \|g\|_{p'}$, $1\le p\le\infty$

The dual $(L^p)^*$ of L^p is isometric to $L^{p'}$ for $1\le p<\infty$ and we have

$$\|f\|_{L^p} = \inf_{\|g\|_{p'}=1} \left|\int_X fg \, d\mu\right| \quad \text{for } 1\le p\le\infty$$

(*) $\|f+g\|_p^p = \int |f+g|^p d\mu \le \|f\|_p^p + \|g\|_p^p$; Since $\mathbb{R}_+\to\mathbb{R}_+$, $x\mapsto x^{1/p}$ is convex for $0<p\le 1$, we have (by Jensen) $\left(\frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_p^p\right)^{1/p} \le \frac{1}{2}\|f\|_p + \frac{1}{2}\|g\|_p$, i.e., $\|f+g\|_p \le 2^{\frac{1}{p}-1}(\|f\|_p+\|g\|_p)$

The constant $2^{\frac{1}{p}-1}$ is sharp; consider $f=\mathbb{1}_{[-1,0]}$, $g=\mathbb{1}_{[0,1]}$. Then $\|f+g\| = 2^{1/p} = 2^{\frac{1}{p}-1}(\|f\|_p+\|g\|_p)$

(or more generally any two functions with disjoint supports)

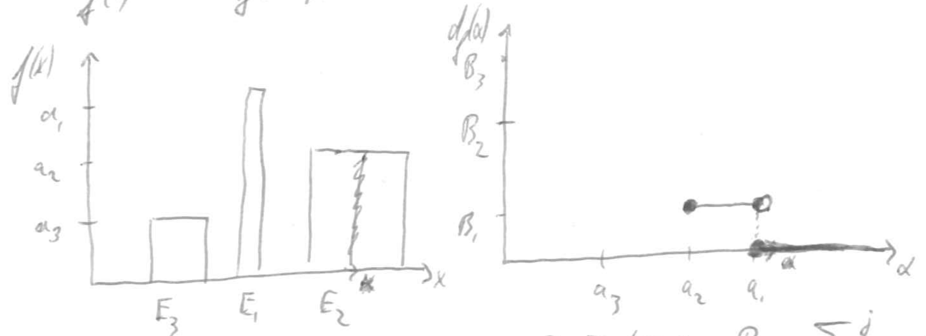
1.1.1 The distribution function

<u>Definition 1.1.1</u> Let f be measurable on X. The distribution function d_f of f ~~defined~~ is defined as

$$d_f : [0,\infty) \to [0,\infty)$$
$$\alpha \mapsto d_f(\alpha) := \mu(\{x \in X : |f(x)| > \alpha\})$$

d_f provides information on the <u>size</u> of f but not on the behavior of f near any given point as d_f is invariant under translations of f, i.e.,

$$d_{f(\cdot)} = d_{f(\cdot - y)}, \quad y \in X$$



$f(x) = \sum_{k=1}^{3} a_k \mathbb{1}_{E_k}(x)$ $\quad$ ~~$B_j = \mu(E_k)$~~ $\quad B_j = \sum_{k=1}^{j} \mu(E_j)$

<u>Example 1.1.2</u> $f(x) = \sum_{k=1}^{N} a_k \mathbb{1}_{E_k}(x), \quad a_1 > a_2 > \dots > a_N, \quad E_k \cap E_j = \emptyset$ for $j \neq k$

$\Rightarrow d_f(\alpha) = 0$ for $\alpha \geq a_1$

$d_f(\alpha) = |E_1|$ for $a_1 > \alpha \geq a_2$

$d_f(\alpha) = |E_1| + |E_2|$ for $a_2 > \alpha \geq a_3$ etc

Denoting $B_j = \sum_{k=1}^{j} |E_j|$, we obtain $d_f(\alpha) = \sum_{j=0}^{N} B_j \mathbb{1}_{[a_{j+1}, a_{j}]}(\alpha)$

where $a_0 = \infty$ and ~~B_{N+1}~~ $B_0 = a_{N+1} = 0$.

<u>Proposition 1.1.3</u> (Simple properties of d_f)
Let f, g be measurable on (X,μ). Then, for all $\alpha, \beta > 0$, we have

(1) $|g| < |f|$ μ-a.e. $\Rightarrow d_g \leq d_f$

(2) $d_{cf}(\alpha) = d_f(\alpha/|c|)$, for $c \in \mathbb{C} \setminus \{0\}$

(3) $d_{f+g}(\alpha + \beta) \leq d_f(\alpha) + d_g(\beta)$

(4) $d_{f \cdot g}(\alpha \cdot \beta) \leq d_f(\alpha) + d_g(\beta)$

<u>Proof</u> Exercise!

Proposition 1.1.4 (Layer cake representation)

For $f \in L^p(X,\mu)$, $0<p<\infty$, we have $\|f\|_{L^p}^p = p\int_0^\infty \alpha^{p-1} d_f(\alpha)\, d\alpha$

Proof Exercise

$$p\int_0^\infty \alpha^{p-1} d_f(\alpha)\,d\alpha = p\int_0^\infty d\alpha\, \alpha^{p-1} \int_X d\mu(x)\, \mathbb{1}_{\{x:|f(x)|>\alpha\}} \qquad \alpha \mapsto \alpha\cdot|f(x)|$$

$$= p\int_X d\mu(x)\, |f(x)|^p \int_0^\infty d\alpha\, \alpha^{p-1} \mathbb{1}_{\{1>\alpha\}} = \int d\mu(x)\, |f(x)|^p \quad \square$$

Corollary $\int_X \varphi(|f|)\, d\mu(x) = \int_0^\infty \varphi'(\alpha)\, d_f(\alpha)\, d\alpha$ whenever $\varphi \in C^1([0,\infty))$ with $\varphi(0)=0$

$$\left(= \int_X d\mu(x) \int_0^1 d\alpha\ \varphi'(\alpha|f|) = \int d\mu(x)\ \varphi(|f|) \right)$$

Definition 1.1.5 (Weak-L^p)

Let $0<p<\infty$ and f measurable on (X,μ). Then $\|f\|_{L^{p,\infty}} = \inf\{C>0 : d_f(\alpha) \le \frac{C^p}{\alpha^p} \text{ for all } \alpha>0\}$

$$= \sup\{\gamma\, d_f(\gamma)^{1/p} : \gamma>0\}$$

Two functions $f,g \in L^{p,\infty}$ are considered to be equal whenever they are μ-a.e. equal to each other.

Some simple properties

- $\|kf\|_{L^{p,\infty}} = |k|\, \|f\|_{L^{p,\infty}}$, $k \in \mathbb{C}$
- $\|f+g\|_{L^{p,\infty}} \le \max\{2, 2^{1/p}\} (\|f\|_{L^{p,\infty}} + \|g\|_{L^{p,\infty}})$ (by Prop. 1.1.3 (3) for $\alpha=\beta=\frac{\alpha}{2}$)
- $\|f\|_{L^{p,\infty}} = 0 \Rightarrow f=0$ μ-a.e.

$\Rightarrow L^{p,\infty}$ quasi-normed linear space for $0<p<\infty$ (later, we show completeness)

Proposition 1.1.6 ($L^p \subseteq L^{p,\infty}$)

Let $0<p<\infty$ and $f \in L^p(X,\mu)$. Then $\|f\|_{L^{p,\infty}} \le \|f\|_{L^p}$, i.e., $L^p \subseteq L^{p,\infty}$.

Proof This follows from Chebyshev's inequality

$$d_f(\alpha) \le \alpha^{-p} \int_{\{x:|f(x)|>\alpha\}} d\mu(x)\, |f(x)|^p \le \alpha^{-p} \|f\|_p^p$$

and Def 1.1.5, i.e., $\|f\|_{p,\infty} = \sup\{\gamma\, d_f(\gamma)^{1/p} : \gamma>0\}$

$$\le \sup\{\tfrac{\gamma}{\gamma}\|f\|_p : \gamma>0\} = \|f\|_p . \quad \square$$

Remark The inclusion is strict, consider, e.g. $|x|^{-d/p} \in L^{p,\infty}(\mathbb{R}^d)$ since

Exercise $\cancel{d_f(|x|^{-d/p}} \; d_{|x|^{-d/p}}(\alpha) = \int_{\mathbb{R}^d} dx\, \mathbb{1}_{|x|<\alpha^{-p/d}} = \alpha^{-p}\, |\mathbb{S}^{d-1}|/d$ but $|x|^{-d/p} \notin L^q$ for any q

i.e., $\| |x|^{-d/p} \|_{L^{p,\infty}} = |\mathbb{S}^{d-1}|$

1.1.2 Convergence in measure

Definition 1.1.7 Let $(f_n)_{n \in \mathbb{N}}$ be measurable on (X, μ).

f_n is said to converge in measure to f if for all $\epsilon > 0$ there exists ~~$N_0 \in \mathbb{Z}^+$~~ $N_0 \in \mathbb{N}$ s.t.

$$n > N_0 \Rightarrow \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) < \epsilon.$$

or equivalently: $\forall \epsilon > 0$: $\lim_{n \to \infty} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) = 0$.

Remark 1.1.8 Clearly, the latter implies the former.

Exercise: show the converse

Proposition 1.1.9 (Conv. in $L^p \Rightarrow$ Conv. in $L^{p,\infty} \Rightarrow$ Conv. in measure)

Let $0 < p \leq ~~\infty~~ \infty$, $f_n, f \in L^{p,\infty}(X, \mu)$

(1) If $f_n, f \in L^p$, ~~then~~ and $f_n \xrightarrow{L^p} f \Rightarrow f_n \xrightarrow{L^{p,\infty}} f$

(2) If $f_n \xrightarrow{L^{p,\infty}} f \Rightarrow f_n \xrightarrow{\text{in measure}} f$

Proof Exercise

Example 1.1.10 Fix $0 < p < \infty$ and define

$$f_{k,j} : [0,1] \to \mathbb{R}_+$$
$$x \mapsto k^{1/p} \mathbb{1}_{(\frac{j-1}{k}, j/k)}(x)$$

for $k \geq 1$ and $1 \leq j \leq k$.

Consider the sequence $\{f_{11}, f_{21}, f_{22}, f_{31}, f_{32}, f_{33}, \dots\}$

Since $|\{x \in [0,1] : f_{kj}(x) > 0\}| = 1/k$, $f_{k,j} \xrightarrow{\text{in measure}} 0$

On the other hand $\|f_{k,j}\|_{L^{p,\infty}} = \sup_{\alpha > 0} \alpha |\{x : f_{kj}(x) > \alpha\}|^{1/p} \geq \frac{1}{2}$ $f \overset{L^{p,\infty}}{\not\to} 0$.

$$\sup_{\alpha > 0} \alpha \left(\int_0^1 dx \, \mathbb{1}_{\{k^{1/p} \mathbb{1}_{(\frac{j-1}{k}, \frac{j}{k})} > \alpha\}}(x) \right)^p = \alpha \int_{\frac{j-1}{k}}^{j/k} ds \, \Theta(k^{1/p} - \alpha)^p = \frac{\alpha \Theta (k^{1/p} - \alpha)}{k^{1/p}}$$

$$\geq \frac{1}{2}$$

$$\alpha = k^{1/p}/2$$

It's well known that convergent sequences in L^p have subsequences that converge μ-a.e. In fact, this can be strengthened.

Thm 1.1.11 Let $(f_n)_{n\in\mathbb{N}}$ and f be measurable on (X,μ) and assume $f_n \xrightarrow{\text{in measure}} f$. Then $f_{n_k} \to f$ μ-a.e. for some subsequence f_{n_k}

Proof For all $k\in\mathbb{N}$ choose n_k inductively s.t.

$(*)$ $\mu(\{x\in X: |f_{n_k}(x)-f(x)|>2^{-k}\})<2^{-k}$ and such that $n_1<n_2<\dots<n_k<\dots$

Define the sets $A_k=\{x\in X: |f_{n_k}(x)-f(x)|>2^{-k}\}$ $\quad (A_1\subseteq A_2\subseteq\dots\subseteq A_\infty)$

By $(*)$, we have $\mu\Big(\underbrace{\bigcup_{k=m}^{\infty} A_k}_{\text{sequence in } m_\infty}\Big)\le\sum_{k\ge m}\mu(A_k)\le\sum_{k\ge m}2^{-k}=2^{1-m}$, $m\in\mathbb{N}$.

and ~~therefore~~ in particular $\mu\Big(\bigcup_{k=1}^{\infty}A_k\Big)\le 1<\infty$.

$$\Rightarrow \underbrace{\mu\Big(\bigcap_{m=1}^{M}\bigcup_{k=m}^{\infty}A_k\Big)}_{\substack{\text{sequence of measures}\\ \text{of the sets } \{\bigcup_{k=m}^{\infty}A_k\}_{m=1}^{M}}} \xrightarrow{M\to\infty} 0$$

The assertion follows from the observation that the null set $\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty}A_k$ contains the set of all $x\in X$ for which $f_{n_k}(x)$ does not converge to $f(x)$. $\square$

In many situations we're given a sequence of functions and we would like to extract a convergent subsequence. This is the content of the following theorem which is a variant of Thm 1.1.11

Definition 1.1.12 (Cauchy in measure)
Let $(f_n)_{n\in\mathbb{N}}$ be measurable in (X,μ). Then $\{f_n\}_{n\in\mathbb{N}}$ is said to be a Cauchy sequence in measure $:\iff \forall\epsilon>0\ \exists N_0\in\mathbb{N}$ s.t. for $n,m>N_0$ we have $\mu(\{x: |f_n(x)-f_m(x)|>\epsilon\})<\epsilon$.

Thm 1.1.13 Let $(f_n)_{n\in\mathbb{N}}$ be measurable in (X,μ) and be Cauchy in measure.
$\Rightarrow (f_n)_{n\in\mathbb{N}}$ has a subsequence that converges μ-a.e.

Proof Similar to Thm 1.1.11 $\to$ exercise.

1.1.3. Generalized Hölder inequality

It's well known that $L^r \subset L^p \cap L^q$ for all $p<r<q$ by Hölder. We have the following sharpening of this result.

Thm 1.1.14 Let $0<p<q\le\infty$ and $f\in L^{p,\infty}\cap L^{q,\infty}$

$$\Rightarrow f\in L^r,\quad r\in(p,q) \text{ and}$$

$$\|f\|_r \le \left(\frac{r}{r-p}+\frac{r}{q-r}\right)^{1/r} \|f\|_{p,\infty}^{(\frac{1}{r}-\frac{1}{q})/(\frac{1}{p}-\frac{1}{q})} \|f\|_{q,\infty}^{(\frac{1}{p}-\frac{1}{r})/(\frac{1}{p}-\frac{1}{q})}$$

with the suitable interpretation when $q=\infty$.

Proof We begin with $q<\infty$, observe $d_f(\alpha)\le\min\left\{\frac{\|f\|_{p,\infty}^p}{\alpha^p}, \frac{\|f\|_{q,\infty}^q}{\alpha^q}\right\}$, and set $B:=\left(\frac{\|f\|_{q,\infty}^q}{\|f\|_{p,\infty}^p}\right)^{1/(q-p)}$. By Prop. 1.1.4 (layer cake),

$$\|f\|_r^r = r\int_0^\infty \alpha^{r-1} d_f(\alpha)\,d\alpha \le r\int_0^\infty \alpha^{r-1}\min\left\{\frac{\|f\|_{p,\infty}^p}{\alpha^p}, \frac{\|f\|_{q,\infty}^q}{\alpha^q}\right\}$$

$$= r\int_0^B \alpha^{r-1-p}\|f\|_{p,\infty}^p\,d\alpha + r\int_B^\infty \alpha^{r-1-q}\|f\|_{q,\infty}^q\,d\alpha$$

$= \dots$ assertion (Note that the integrals converge since $r-p>0$, $r-q<0$)

For $q=\infty$ we merely use $d_f(\alpha)\le\alpha^{-p}\|f\|_{p,\infty}^p$ ~~and~~ ~~for~~ for $\alpha\le\|f\|_{L^\infty}$

$(d_f(\alpha)=0$ when $\alpha>\|f\|_\infty)$

and thus, we only need to consider the first summand in the above integrals. We obtain $\|f\|_r^r\le\frac{r}{r-p}\|f\|_{p,\infty}^p\|f\|_\infty^{r-p}$ ☐

~~Let us finally summariz~~

We conclude by noting (recalling) some convolution inequalities

$\|f*g\|_r \le \|f\|_p\|g\|_q$ (Young) $\quad 1\le p,q,r\le\infty,\ 1+\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$

$\|f*g\|_r \lesssim_{p,q} \|f\|_p\|g\|_{q,\infty}$ (generalized Young) $\quad 1<p,q,r<\infty$

$\|f*g\|_{r,\infty} \lesssim_{p,q} \|f\|_{p,\infty}\|g\|_{q,\infty}$ (weak Young) $\quad 1<p,q,r<\infty$

$\llcorner$ Reed-Simon 2, p.32 or Grafakos p.21 where $\|f\|_{p,\infty}$ is replaced $\|f\|_p$

~~Proof of Corollary 1.2.2'' for~~ q_1 $0<p_i<q_i$ ~~$<\infty$, $q_0 \le p_1$, $q_0<q_1$~~

~~following~~

~~Proof of Thm 1.2.2' for $1\le r<\infty$, $q_0>1$,~~ T sublinear ~~following Tao's~~

~~notes (Lecture 1, Thm 8.5~~

~~on harmonic analysis~~

1.2 Lorentz spaces

Folland - Real Analysis (Section 6.4)
Adams - Fournier - Sobolev Spaces (p. 221)
notes by G. Rey

Suppose f is measurable on (X,μ). In the following, we construct the spherically decreasing rearrangement f^* of f, defined on $[0,\infty)$ and characterized by $d_f(\alpha)=d_{f^*}(\alpha)$.

1.2.1 Decreasing rearrangements

Definition 1.2.1 The decreasing rearrangement of a complex-valued, measurable f is the function f^* defined on $[0,\infty)$ by

$$f^*(t):=\inf\{s>0: d_f(s)\le t\}$$

($d_f(s)=\mu(\{x\in X: |f(x)|>s\})$ is decreasing in s)

with the convention $\inf\emptyset=\infty$, i.e. $f^*(t)=\infty$ whenever $t<d_f(\alpha)$ $\forall \alpha\ge 0$.

Clearly, f^* is decreasing and supported on $[0,\mu(X)]$. ($d_f(s)\le \underset{d_f(s=0)}{\mu(X)}$ and when $t>\mu(X)$, then we can take $s=0$)

Let's consider some examples.

Example 1.2.2 $f(x)=\sum_{j=1}^N a_j \mathbb{1}_{E_j}(x)$ where E_j have finite measure and are pairwise disjoint, and $a_1>a_2>\ldots>a_N$. $d_f(\alpha)=\sum_{j=0}^N B_j \mathbb{1}_{[a_{j+1},a_j)}(\alpha)$

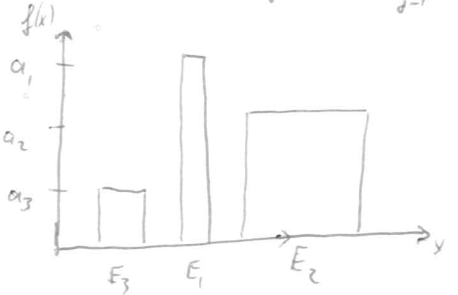
We already saw in Example 1.1.2 that ~~$d_f(\alpha)=\sum_{j=0}^N B_j \mathbb{1}_{[a_j, a_{j+1})}(\alpha)$~~

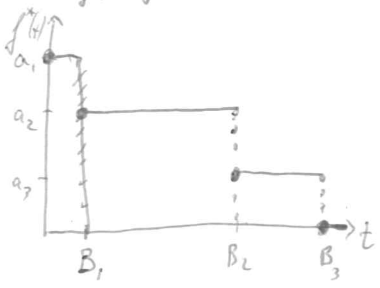
where $B_j=\sum_{k=1}^j \mu(E_k)$ and $a_{N+1}=B_0=0$, $a_0=\infty$.

Now, for $B_0\le t<B_1$ the smallest $s>0$ with $d_f(s)\le t$ is a_1

Similarly, for $B_1\le t<B_2$, — " — is a_2

etc $\Rightarrow$ $f^*(t)=\sum_{j=1}^N a_j \mathbb{1}_{[B_{j-1},B_j)}(t)$





<u>Example 1.23</u> Consider $f: \mathbb{R}^d \to \mathbb{R}_+$
(Exercise) $x \mapsto f(x) = \frac{1}{1+|x|^p}$ $\quad 0<p<\infty$

Then, one can computes $d_f(\alpha) = \begin{cases} |S^{d-1}| \left(\frac{1}{\alpha}-1\right)^{d/p} & \alpha<1 \\ 0 & \alpha \geq 1 \end{cases}$

and therefore $f^*(t) = \frac{1}{1+(t/|S^{d-1}|)^{p/d}}$

<u>Example 1.24</u> $g: \mathbb{R}^d \to \mathbb{R}$ $\quad d_g(\alpha) = \begin{cases} 0 & \alpha \geq 1 \\ \infty & \alpha < 1 \end{cases} \rightarrow g^*(t) = 1, \ t \geq 0$
$x \mapsto 1-e^{-|x|^2}$

Interpretation: Although quantitative information is preserved, significant qualitative information is lost when passing to its decreasing rearrangement.

<u>Proposition 1.25</u> Let f, g, f_n be μ-measurable, $k \in \mathbb{C}$, $0<t, s, t_1, t_2<\infty$

(0) f^* is right-continuous and decreasing

~~(1) $d_f(f^*(t)) \leq t$~~

(1) $f^*(d_f(\alpha)) \leq \alpha, \ \alpha>0$

(2) $d_f(f^*(t)) \leq t$

(3) $f^*(t) > s \iff t < d_f(s)$, i.e., $\{t \geq 0 \ f^*(t)>s\} = [0, d_f(s))$

(4) $|g| \leq |f| \ \mu$-a.e. $\Rightarrow g^* \leq f^*$ and $|f|^* = f^*$

(5) $(kf)^* = |k| f^*$

(6) $(f+g)^*(t_1+t_2) \leq f^*(t_1) + g^*(t_2)$

(7) $(fg)^*(t_1+t_2) \leq f^*(t_1) \cdot g^*(t_2)$

(8) $|f| \leq \underline{\lim}_{n\to\infty} |f_n| \ \mu$-a.e. $\Rightarrow f^* \leq \underline{\lim}_{n\to\infty} f_n^*$

(9) $|f_n| \nearrow |f| \ \mu$-a.e $\Rightarrow f_n^* \nearrow f^*$

(10) $t \leq \mu(\{|f| \geq f^*(t)\})$ if $\mu(\{|f| > f^*(t) - c\}) < \infty$ for some c

(11) $d_f = d_{f^*}$

(12) $(|f|^p)^* = (f^*)^p, \ 0<p<\infty$

(13) $\int_X |f|^p d\mu = \int_0^\infty f^*(t)^p dt, \ 0<p<\infty$

(14) $\|f\|_{L^\infty} = f^*(0)$

(15) $\sup_{t>0} t^q f^*(t) = \sup_{\alpha>0} \alpha (d_f(\alpha))^q, \ 0<q<\infty$ $\quad$ <u>Proof</u> Exercise (Grafakos)

1.2.2 Lorentz spaces

Definition 1.2.6 Let f be measurable on (X,μ), $0<p,q\le\infty$, and define

$$\|f\|_{L^{p,q}(X)} = \begin{cases} \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q} = \| t^{1/p} f^*\|_{L^q(\mathbb{R}_+;\frac{dt}{t})} & q<\infty \\ \sup_t t^{1/p} f^*(t) \overset{\text{Prop 1.1.5}}{=} \sup_{\alpha>0} \alpha\, d_f(\alpha)^{1/p} & q=\infty \end{cases}$$

Clearly, by the previous definitions $L^{\infty,\infty}=L^\infty$ and $L^{p,\infty}=$ weak L^p from ~~the first lecture~~. ~~Section 1~~. Def 1.1.5
(+Prop 1.2.5) $L^{p,p}=L^p$ (isometrically) (+ Prop 1.2.5 (13))

Moreover, observe $\| |g|^r\|_{L^{p,q}} = \|g\|^r_{L^{pr,qr}}$, $0<p,r<\infty$, $0<q<\infty$

and $\|f(\lambda x)\|_{L^{p,q}} = \lambda^{-d/p}\|f\|_{L^{p,q}}$ (b/c $d_{f(\lambda\cdot)}(\alpha)=\lambda^{-d}d_f(\alpha)$ and $(f(\lambda\cdot))^* = f^*(\lambda^d\cdot)$)

(Exercise!)

Proposition 1.2.7 For $0<p<\infty$ and $0<q\le\infty$, we have $\|f\|_{L^{p,q}} = p^{1/q}\left(\int_0^\infty [d_f(s)^{\frac1p}s]^q\frac{ds}{s}\right)^{1/q}$

Proof $q=\infty$ follows from Prop. 1.2.5 (15)
$q<\infty$: Consider the simple fct $f(x)=\sum_{j=1}^N a_j \mathbb{1}_{E_j}(x)$ from Example 1.1.2

$$\Rightarrow d_f(s)=\sum_{j=1}^N B_j \mathbb{1}_{[a_{j+1},a_j)}(s),\quad a_{N+1}=B_0=0$$

$$f^*(t)=\sum_{j=1}^N a_j \mathbb{1}_{[B_{j-1},B_j)}(t)$$

$$\overset{\text{Def 1.3.6}}{\Rightarrow} \|f\|_{p,q} = \left(\frac{p}{q}\right)^{1/q}\left[a_1^q B_1^{q/p} + a_2^q\left(B_2^{\frac qp}-B_1^{\frac qp}\right)+\dots+a_N^q\left(B_N^{\frac qp}-B_{N-1}^{\frac qp}\right)\right]^{\frac1q}$$

(and $\|f\|_{p,\infty} = \sup_{1\le j\le N} a_j B_j^{1/p}$)

this is just the assertion

For general f, we find a sequence of non-negative simple functions s.t. $f_n \nearrow |f|$
Then $d_{f_n}\nearrow d_f$ (Exercise) and $f_n^*\nearrow f$ (Prop 1.3.5 (9)) and the assertion follows from monotone convergence thm.

Prop. 1.2.8 Suppose $0<p\le\infty$, $0<q<r\le\infty$. Then $L^{p,q}\subseteq L^{p,r}$, i.e. $\|f\|_{L^{p,r}}\lesssim_{pqr}\|f\|_{L^{p,q}}$

Proof $p=\infty$ trivial; $p<\infty$: $t^{1/p}f^*(t) = \left\{\frac qp\int_0^t [s^{1/p}f^*(t)]^q\frac{ds}{s}\right\}^{1/q}$

$$\overset{f^*\text{ decreasing}}{\le} \left\{\frac qp\int_0^t [s^{1/p}f^*(s)]^q\frac{ds}{s}\right\}^{1/q} \le \left(\frac qp\right)^{1/q}\|f\|_{L^{p,q}}.$$

Now, taking $\sup_{t>0}$, we obtain $\|f\|_{p,\infty}\le\left(\frac qp\right)\|f\|_{p,q}$, i.e., we're done when $r=\infty$.

When $r<\infty \Rightarrow \|f\|_{p,r} = \left\{\int_0^\infty (t^{1/p}f^*(t))^{r-q+q}\frac{dt}{t}\right\}^{1/r} \le \underbrace{\|f\|_{p,\infty}^{(r-q)/r}}_{\lesssim \|f\|_{p,q}} \|f\|_{p,q}^{q/r}$ $\square$

$L^{p,q}$ Banach ?

Unfortunately, we only have the quasi-triangle inequality for $L^{p,q}$.

Counterexample $f(t) = t$, $g(t) = 1-t$ on $[0,1] \Rightarrow f^*(\alpha) = g^*(\alpha) = (1-\alpha)\,\mathbb{1}_{[0,1]}(\alpha)$ and the usual triangle inequality wrt $\|\cdot\|_{p,q}$ would be equivalent to $\frac{p}{q} \leq 2^q \frac{\Gamma(q+1)\,\Gamma(q/p)}{\Gamma(q+1+q/p)}$

which fails in general.

~~However,~~

However, since $(f+g)^*(t) \leq f^*(t/2) + g^*(t/2)$ and $\|f+g\|_{p,\infty} \leq \max\{1, 2^{1/p}\}(\|f\|_{p,\infty} + \|g\|_{p,\infty})$, we have $\|f+g\|_{p,q} \leq 2^{1/p} \max\{1, 2^{(1-q)/q}\}(\|f\|_{p,q} + \|g\|_{p,q})$.

Moreover, $\|f\|_{p,q} = 0 \Rightarrow f = 0$ μ-a.e., i.e., $L^{p,q}$ is a quasi-normed linear space.
In fact $L^{p,q}$ is complete

<u>Theorem 1.2.9</u> Let $0 < p,q \leq \infty$. Then $L^{p,q}(X,\mu)$ are quasi-Banach spaces.
If $p,q > 1$, they are Banach.

<u>Proof</u> Exercise (Grafakos, Th 1.4.11)

<u>Theorem 1.2.10</u> Simple functions are dense in $L^{p,q}$ when $0 < p,q < \infty$. (Exercise!) (Grafakos Th 1.4.13)

~~Theorem 1.3.11~~

<u>Remark</u> This fails when $q = \infty$ for all $0 < p \leq \infty$. ~~Afr~~

<u>Proposition</u> 1.2.11 (Rey's notes) Monotone convergence: $|f_n| \nearrow f$ μ-a.e. $\Rightarrow \|f\|_{p,q} = \lim \|f_n\|_{p,q}$
Fatou $\|\lim_{n\to\infty} f_n\|_{p,q} \leq \underline{\lim}_n \|f_n\|_{p,q}$

We will now give some alternative characterizations of $L^{p,q}$ (following Tao's notes on Fourier analysis, lecture 1, section 6)

<u>Definition 1.2.12</u> (1) A <u>sub-step function of height H and width W</u> is any function f supported on a set E with bounds $|f(x)| \leq H$ a.e. and $\mu(E) \leq W$ (Thus, $|f| \leq H\,\mathbb{1}_E$)

(2) A quasi-step function of height H and width W is any function f supported on a set E with bounds $|f(x)| \sim H$ a.e. on E and $\mu(E) \sim W$ (Thus $|f| \sim H\,\mathbb{1}_E$)

Remark 1.2.13 From the binary ~~decomposition~~ expansion of $[0,1]$, one sees that any non-negative sub-step fct f of height 1 and width W can always be decomposed as $\sum_{k\geq 1} 2^{-k} f_k$ where f_k is an actual step function of height 1 and width of at most W. By homogeneity, we have a similar statement for other heights
$\Rightarrow$ bounds on step functions extend to bounds on sub-step fcts (and hence quasi-step fcts)
$\Rightarrow$ $\|\text{sub-step-fct}\|_{p,q} = O_{p,q}(H \cdot W^{1/p})$
$\|\text{quasi-step-fct}\|_{p,q} \sim_{p,q} H \cdot W^{1/p}$

In the converse direction it turns out that every $L^{p,q}$ fct can be decomposed as an ℓ^q sum of "very different" $L^{p,q}$-normalized sub-step or quasi-step fcts

Theorem 1.2.14 (Characterizations of $L^{p,q}$) (Tao's Fourier analysis notes, Thm 6.6 in lecture!)
Let f be a fct, $0<p<\infty$, $1\leq q\leq\infty$, and $0<A<\infty$. Then the following are equivalent up to changes of the involved implied constants

(i) $\|f\|_{p,q} \lesssim_{p,q} A$

vertically dyadic decomp.

(ii) There exists a decomposition $f=\sum_{m\in\mathbb{Z}} f_m$ where each f_m is a quasi-step fct of height 2^m and some width $W_m \in (0,\infty)$. The f_m have disjoint supports and $\|2^m W_m^{1/p}\|_{\ell^q_m(\mathbb{Z})} \lesssim_{p,q} A$
$\uparrow$ ℓ^q-summation with respect to m

(iii) There exists a pointwise bound $|f| \leq \sum_{m\in\mathbb{Z}} 2^m \mathbb{1}_{E_m}$ with $\|2^m \mu(E_m)^{\frac{1}{p}}\|_{\ell^q_m(\mathbb{Z})} \lesssim_{p,q} A$

horizontally dyadic decomp.

(iv) There exists a decomposition $f=\sum_{n\in\mathbb{Z}} f_n$ where ~~fn~~ each f_n is a sub-step function of some height $0<H_n<\infty$ and width 2^n whose supports are pairwise disjoint. Moreover, H_n are non-increasing in n,
$H_{n+1} \leq |f_n| \leq H_n$ on supp f_n and
$\|H_n 2^{n/p}\|_{\ell^p_n(\mathbb{Z})} \lesssim_{p,q} A \quad (*)$

(v) There exists a pointwise bound $|f| \leq \sum_{n\in\mathbb{Z}} H_n \mathbb{1}_{E_n}$ where $\mu(E_n) \lesssim_{p,q} 2^n$ and $(*)$ holds

Remarks 1.2.15 (ii), (iv) are useful when trying to use an $L^{p,q}$ bd. on f
(iii), (v) are useful when trying to obtain — " —

Heuristically: If f is a quasi-step fct of height H and width W, then $\|f\|_{p,q} \sim_{p,q} HW^{1/p}$. But, if f is a sum $\sum_n f_n$ of quasi-step fcts f_n and H_n or W_n are sufficiently variable in n, then $\|\sum f_n\|_{p,q} \sim_{p,q} \|H_n W_n^{1/p}\|_{\ell^q_n}$ (if one or the other grows like 2^n)

Remark 1.2.16 Suppose there's an N s.t. $A \le |f(x)| \le A \cdot N$ for some A, i.e.,
N is the ratio between the tallest and lowest non-zero height of f.
Then the above Thm shows that the norms $\|f\|_{p,q_1}$, $\|f\|_{p,q_2}$ only differ by multiplicative powers of $\underline{\log N}$.
Similarly, if the broadest width and the narrowest width of f differ by N (i.e., if $\mu(X)$ equals N times the granularity c of X).
This means that the secondary exponent in $L^{p,q}$ (i.e. q) only offers logarithmic correction to the L^p-norms.
On the other hand $\|\sum_{n=1}^N f_n\|_p \le N^{\frac{1}{p}-1}\sum_{n=1}^N \|f_n\|_p$ for $0<p<1$ shows that varying the primary exponent p leads to polynomially strong changes in the norm.

We conclude the discussion about Lorentz spaces with two important consequences of the above Thm.

Theorem 1.2.17 (Hölder inequality in $L^{p,q}$; due to O'Neil)
If $0<p_1,p_2,p<\infty$ and $0<q_1,q_2,q\le\infty$ obey $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$, then
$\|f\cdot g\|_{p,q} \lesssim_{p_1,p_2,p,q_1,q_2,q} \|f\|_{p_1,q_1}\|g\|_{p_2,q_2}$ whenever the norms on the right sides are finite.

~~Proof~~
Remark There's also a Young-type inequality (O'Neil)
$$\|f*g\|_{L^{r,s}} \lesssim \|f\|_{p,s_1}\|g\|_{q,s_2}, \quad 1<p,q,r<\infty, \; 0<s_1,s_2<\infty, \quad 1+\frac{1}{r}=\frac{1}{p}+\frac{1}{q}, \; \frac{1}{s}=\frac{1}{s_1}+\frac{1}{s_2}$$

Proof In the following, we assume $\|f\|_{p_1,q_1}=\|g\|_{p_2,q_2}=1$ and drop the dependence on the parameters for brevity.
By (i)$\Leftrightarrow$(v) in Thm 1.2.14, we may ~~denote~~ estimate $|f|\le\sum_n H_n 1\!\!1_{E_n}$ and $|g|\le\sum_n H_n' 1\!\!1_{E_n'}$ where $\mu(E_n),\mu(E_n')\lesssim 2^n$ and
$\|H_n 2^{n/p_1}\|_{\ell_n^{q_1}}$, $\|H_n' 2^{n/p_2}\|_{\ell_n^{q_2}} \lesssim 1$.

$\Rightarrow |fg|\le\sum_{n,k} H_n H_{n+k}' 1\!\!1_{E_n\cap E_{n+k}'}$. By the quasi triangle inequality (Exercise!)
$\|\sum_{n=1}^N f_n\|_p \le N^{\frac{1}{p}-1}\sum_n \|f_n\|_p$
and monotonicity, it suffices to prove $\|\sum_{k\ge0}\sum_n H_n H_{n+k}' 1\!\!1_{E_n\cap E_{n+k}'}\|_{p,q}\lesssim 1$
and $\|\sum_{k<0}\sum_n$ — " — $\|_{p,q}\lesssim 1$
By symmetry, it suffices to consider the $k\ge0$ series

To prove $\| \sum_{k \geq 0} \sum_n H_n H'_{n+k} 1_{E_n \cap E'_{n+k}} \|_{p,q} \lesssim 1$, $(*)$

observe that $E_n \cap E'_{n+k}$ has measure at most 2^n (recall $\mu(E_n), \mu(E'_n) \lesssim 2^n$)

~~so by the equivalence (i) ⇔ (v) in Thm 1.2.14 for f, g, we have~~ ~~$\|\sum$ (with $\sum_{k\geq 0}$ this is~~

~~$(*)$ is equivalent to~~ $\cancel{\| \sum_n \sum_{k \geq 0} H_n H'_{n+k} 1_{E_n \cap E_{n+k}} \|_{p,q} \lesssim \| \sum_{k\geq 0} H_n H'_{n+k} 2^{n/p} \|_{\ell^q_n}}$.

Now consider fixed k. Suppose we could prove $\| \sum_n H_n H'_{n+k} 1_{E_n \cap E'_{n+k}} \|_{p,q} \lesssim 2^{-k/p_2}$,

then we were done (Exercise! see also Q1 in Lecture 1 in Tao's notes or Lemma 5.1 in Rey's notes)

But since $\mu(E_n \cap E'_{n+k}) \lesssim 2^n$, we can apply the equivalence (i) ⇔ (v) of Thm 1.2.14 which says $\| \sum_n H_n H'_{n+k} 1_{E_n \cap E'_{n+k}} \|_{p,q} \lesssim \| H_n H'_{n+k} 2^{n/p} \|_{\ell^q_n}$.

But by the ordinary Hölder ineq. (for ℓ^q, $0<q<\infty$) $\| H_n H'_{n+k} 2^{n/p} \|_{\ell^q_n} \lesssim \| H_n 2^{n/p_1} \|_{\ell^{q_1}_n} \| H'_{n+k} 2^{n/p_2} \|_{\ell^{q_2}_n}$

$= \text{—''—} \cdot \| H'_n 2^{n/p_2} \|_{\ell^{q_2}_n} \cdot 2^{-k/p_2}$

$\cancel{|u_n v_n|^q \leq |u_n|^{q/q_1} |v_n|^{q_2/q}}$

$|u_n v_n|^q \leq \frac{|u_n|^{q_1}}{q_1} + \frac{|v_n|^{q_2}}{q_2}$

by shifting the second n-summation ☐

Theorem 1.2.18 (Dual characterization of $L^{p,q}$)

Let $1<p<\infty$ and $1 \leq q \leq \infty$. Then for any $f \in L^{p,q}$,

$$\|f\|_{L^{p,q}} \sim_{p,q} \sup\{ |\int_X f \bar{g} \, d\mu| : \|g\|_{L^{p',q'}} \leq 1 \}.$$

More precisely, if $q = \infty$, then $\|f\|_{L^{p,\infty}} \sim_p \sup\{ \mu(E)^{-\frac{1}{p'}} |\int_X f(x) 1_E(x) \, d\mu(x)| : 0 < \mu(E) < \infty \}$.

~~where also p=1 is admissible.~~

Proof. To obtain "$\gtrsim_{p,q}$", we simply estimate

$$|\int_X f \bar{g} \, d\mu| \leq \|fg\|_{L^1} = \|fg\|_{L^{1,1}}$$

and use Thm 1.2.17 (Hölder). To obtain "$\lesssim_{p,q}$", we normalize $\|f\|_{p,q} = 1$ and by homogeneity it suffices to find g with $\|g\|_{p',q'} \leq 1$ and $\int_X f \bar{g} \, d\mu(x) \gtrsim 1$.

We start with $q=\infty$ and assume $f \geq 0$.

Let $E_\lambda = \{x \in X : f(x) > \lambda\}$ which satisfies $\mu(E_\lambda) < \infty$ since $f \in L^{p,\infty}$.

If $\mu(E_\lambda) \neq 0$, then

$$\lambda\, \mu(E_\lambda)^{1/p} = \frac{1}{\mu(E_\lambda)^{1/p'}} \cdot \lambda \mu(E_\lambda) \leq \frac{1}{\mu(E_\lambda)^{1/p'}} \int_X f(x) \mathbb{1}_{E_\lambda}\, d\mu \leq \sup_{0<\mu(E)<\infty} \frac{1}{\mu(E)^{1/p'}} \left| \int_X f(x) \mathbb{1}_E\, d\mu \right|$$

Taking $\sup_{\lambda>0}$ yields the assertion.

Now suppose f is not necessarily non-negative. Then decompose $f = u_0^+ - u_0^- + i(u_1^+ - u_1^-)$ where $u_i^\pm \geq 0$ for $i=0,1$, so by the quasi-triangle inequality it suffices to show $\|u_i^\pm\|_{L^{p,\infty}} \leq \sup_{0<\mu(E)<\infty} \frac{1}{\mu(E)^{1/p'}} \left| \int_X f(x) \mathbb{1}_E(x)\, d\mu(x) \right|$.

But since $u_i^\pm = e^{i\theta} f \mathbb{1}_F$ for some $F \subseteq X$, we have for any given set E with $0 < \mu(E) < \infty$

$$\frac{1}{\mu(E)^{1/p'}} \left| \int_X u_i^\pm \mathbb{1}_E\, d\mu \right| = 0 \leq \sup_{0<\mu(E')<\infty} \frac{1}{\mu(E')^{1/p'}} \left| \int_X f \mathbb{1}_{E'}\, d\mu \right|, \quad \mu(E \cap F) = 0$$

respectively
$$\frac{1}{\mu(E)^{1/p'}} \left| \int_X u_i^\pm \mathbb{1}_E\, d\mu \right| = \frac{1}{\mu(E)^{1/p'}} \left| \int_X e^{i\theta} f \mathbb{1}_{E\cap F}\, d\mu \right|$$
$$\leq \frac{1}{\mu(E\cap F)^{1/p'}} \left| \int_X f \mathbb{1}_{E\cap F}\, d\mu \right| \leq \sup_{0<\mu(E')<\infty} \frac{1}{\mu(E')^{1/p'}} \left| \int_X f \mathbb{1}_{E'}\, d\mu \right|$$

when $\mu(E \cap F) > 0$

This concludes the proof for ~~gell.~~ $q=\infty$.

~~Now, we consider " $\lesssim_{p,q}$ " for $q<\infty$ and first restrict ourselves to $g \geq 0$, a simple function with support of finite measure. (Exercise → Thm 6.12 in Tao's notes)~~

$q < \infty \Rightarrow$ exercise (see Tao's notes, Thm 6.12)

~~For general~~

Remark: For $1<p<\infty$, $1 \leq q \leq \infty$ one also has the alternative dual characterization

$$\|f\|_{L^{p,q}} \sim_{p,q} \sup\left\{ \left| \int_X f g\, d\mu \right| : g \in \Sigma,\ \|g\|_{L^{p',q'}} \leq 1 \right\}$$

where Σ is the set of all finite combinations of characteristic functions of sets of finite measure.
(see also Corollary 3.4 in Rey's notes)

$T(f+g) = Tf + Tg\,, \quad T(\lambda f) = \lambda Tf$

1.3 Interpolation

Assume (X,μ) and (Y,ν) are two measure spaces
T a linear operator, initially defined on the set of really simple functions $f = \sum_{k=1}^{N} a_k \mathbb{1}_{E_k}$ on X s.t. Tf is a ν-measurable fct on Y.

Let $0 < p,q < \infty$ and assume $\exists\, C_{pq} < \infty$ s.t. $\|Tf\|_{L^q(Y,\nu)} \le C_{pq} \|f\|_{L^p(X,\mu)}$
then, by density, T admits a unique bounded extension from $L^p(X,\mu)$ to $L^q(Y,\nu)$
which is (in abuse of notation) also denoted by T. In this case, T is said to be of <u>strong type $(L^p(X,\mu), L^q(Y,\nu))$</u> or simply (p,q).
If the above holds for $L^{q,\infty}(Y,\nu)$ instead, then T is said to be of <u>weak-type (p,q)</u> (i.e. $\sup_{\gamma>0} \gamma\, d_{Tf}(\gamma)^{1/q} \le C_{pq} \|f\|_p$

or $\quad d_{Tf}(\alpha) \le C_{pq} \cdot \alpha^{-q} \|f\|_p^q \quad \forall\, \alpha > 0$

1.3.1 Real interpolation (Marcinkiewicz)

<u>Def 1.3.1</u> T is said to be <u>sublinear</u> if $|T(f+g)| \le |Tf| + |Tg|$ and $|T(\lambda f)| = |\lambda| |Tf|$
<u>quasilinear</u> if $\exists\, K > 0$ s.t. $|T(f+g)| \le K(|Tf| + |Tg|)$ and $|T(\lambda f)| = |\lambda| |Tf|$

<u>Theorem 1.3.2</u> Let $0 < p_0 < p_1 \le \infty$ and T be a sublinear operator defined on the space of measurable functions $L^{p_0}(X,\mu) + L^{p_1}(X,\mu)$, taking values in the space of measurable functions on Y. Assume $\exists\, A_0, A_1 > 0$ s.t.

$$\|Tf\|_{L^{p_0,\infty}(Y)} \le A_0 \|f\|_{L^{p_0}(X)}\,, \quad f \in L^{p_0}(X)$$
$$\|Tf\|_{L^{p_1,\infty}(Y)} \le A_1 \|f\|_{L^{p_1}(X)}\,, \quad f \in L^{p_1}(X)$$

Then, for all $p_0 < p < p_1$ and all $f \in L^p(X)$, we have

$$\|Tf\|_{L^p(Y)} \le A \|f\|_{L^p(X)}\,, \quad A = 2\left(\frac{p}{p-p_0} + \frac{p}{p_1 - p}\right)^{1/p} \cdot A_0^{(\frac{1}{p} - \frac{1}{p_1})/(\frac{1}{p_0} - \frac{1}{p_1})} A_1^{(\frac{1}{p_0} - \frac{1}{p})/(\frac{1}{p_0} - \frac{1}{p_1})}.$$

<u>Remark</u> The assumptions can be further weakened; in fact, T only needs to satisfy the corresponding <u>restricted weak-type estimates</u>

$\alpha\, d_{T\mathbb{1}_E}(\alpha)^{\frac{1}{q_i}} \lesssim |E|^{1/p_i} \;\forall \alpha$ which is equivalent to $|\langle \mathbb{1}_F, T\mathbb{1}_E \rangle| \lesssim |E|^{\frac{1}{p_i}} |F|^{\frac{1}{q_i'}}$ (with $q_i = p_i$)

<u>Exercise</u>: Show "$\Rightarrow$" in the above formulations of restricted weak-type (see Lecture 2, Lemma 2.2 in Tao's notes on restriction theory)
Show "$\Leftarrow$" by plugging in $F = \{ \mathrm{Re}(T\mathbb{1}_E) > \alpha \}$)

(maybe rather $F = \{ |T\mathbb{1}_E| > \alpha \}$)

Theorem 1.3.2' (Thm 1.4.19 in Grafakos)

Suppose $0 < p_i < q_i \le \infty$ $(i = 0, 1)$ and $0 < r \le \infty$.
Let T be a quasilinear operator defined on $L^{p_0}(X) + L^{p_1}(X)$, taking values in the set of measurable fcts in Y or a sublinear operator ~~taking values in the set of~~ defined on the really simple functions in X and taking values as before.
Assume T is of restricted weak-type (p_i, q_i) with bounds M_0, M_1, i.e. $\|T\mathbb{1}_E\|_{L^{q_i,\infty}} \le M_i |E|^{1/p_i}$ for all measurable $E \subseteq X$.
Fix $\theta \in (0,1)$ and let $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.
Then $\exists\, M = M_{K, p_i, q_i, M_i, r, \theta}$ (K: constant from quasilinearity) such that for all $f \in \mathrm{dom}(T) \cap L^{p,r}(X)$, we have $\|Tf\|_{L^{q,r}} \le M \|f\|_{L^{p,r}}$

Here, $L^{p,r}$ is the Lorentz space, which we will define later. Without proof, we mention $L^{p,\infty} \subseteq L^{p_0} + L^{p_1}$ and due to $L^{p,r} \subseteq L^{p,\infty}$ (later) for $0<r\le\infty$, we see that T is indeed well-defined on $L^{p,r}$ for all $r \le \infty$. If $r<\infty$ and T is linear and defined on the set of really simple fcts in X, then T has a unique extension that satisfies the above assertion for all $f \in L^{p,r}(X)$ since simple fcts are dense in $L^{p,r}$.

Corollary 1.3.2'' Let T be as above and $0 < p_0 < p_1 \le \infty$, $0 < q_0 < q_1 \le \infty$. If T maps $L^{p_i}(X)$ to $L^{q_i,\infty}(Y)$ and $\frac{1}{p_\theta}, \frac{1}{q_\theta}$ are defined as before with $p_\theta \le q_\theta$, then T satisfies $\|Tf\|_{L^{q_\theta}} \le C \|f\|_{L^{p_\theta}}$, $f \in \mathrm{dom}(T)$. Moreover, if T is linear, then it has a ~~unique~~ bounded extension from $L^{p_\theta}(X)$ to $L^{q_\theta}(X)$.

~~Although~~ This Corollary follows immediately from the above Thm (by setting $r = q$ (note $L^{p,q} \supseteq L^{p,p} = L^p$ in the above assertion of Thm 1.2.2'). We will give an ~~independent~~ proof of ~~it~~ Thm 1.3.2' for the special case ~~where~~ h. ~~Proof of~~ where T is sublinear, $1 \le r \le \infty$, $q_\theta > 1$. ~~later on~~. (see also Tao's notes, lecture 1, Thm 8.5)
~~Actually we're going to let T act on ... sub-step fcts of height A and width W~~

Theorem 1.3.3 Let (X,μ) be a measure space, $0<p<\infty$, and $f\in L^{p,\infty}(X)$.
Then the following are equivalent

(1) $\|f\|_{p,\infty} \lesssim_p 1$

(2) For every $E\subseteq X$ with $\mu(E)<\infty$ there is $E'\subseteq E$ such that $\mu(E)\leq 2\mu(E')$ such that $\left|\int_X f(x)\mathbb{1}_{E'}(x)\,d\mu(x)\right| \lesssim_p \mu(E')^{1/p'}$

Proof Without loss of generality we assume $f\geq 0$ and $\|f\|_{p,\infty}=1$ ~~and ...~~. The proof of (2) $\Rightarrow$ (1) is almost identical to the one in Thm 1.2.18, (for $q=\infty$) so let us focus on (1) $\Rightarrow$ (2).

Let $E\subseteq X$ be s.t. $0<\mu(E)<\infty$ and $\alpha=\left(\frac{2}{\mu(E)}\right)^{1/p}$ and define $E':=E\setminus\{x: f(x)>\alpha\}$

Since $\mu(\{x: f(x)>\alpha\}) \leq \frac{\mu(E)}{2}$ we have the estimates $\mu(E')\leq\mu(E)$ and

$\cancel{\alpha^{-p}\mu(\{x: f(x)>\alpha\}) \leq \|f\|_{p,\infty}=1}$ bc $\alpha^{+p}\mu(\{x: f(x)>\alpha\})\leq 1$

$\cancel{\Leftrightarrow \mu(E)\leq \frac{2}{\mu(\{x}}$ $\Leftrightarrow \mu(\dots)\leq\alpha^{-p}=\frac{\mu(E)}{2}$

$\cancel{\frac{\mu(E)}{2}}$ $\mu(E')=\mu(E)-\mu(\{x: f>\alpha\}) \geq \mu(E)-\frac{\mu(E)}{2}=\frac{\mu(E)}{2}$

Finally, we estimate $\int_X f\mathbb{1}_{E'}\,d\mu \leq \alpha\cdot\mu(E')=\left(\frac{2}{\mu(E)}\right)^{1/p}\mu(E')\leq 2^{\frac{1}{p}}\mu(E')^{1/p'}$ $\boxtimes$

on E', we have $f\leq\alpha$ $\qquad$ $\mu(E)\geq\mu(E')$

Let us abbreviate some things.

Definition 1.3.4 Σ^+ ... set of all non-negative simple functions with support of finite measure

Σ_c ... set of all finite combinations of characteristic functions of sets of finite measure

A sublinear operator T is said to be of restricted weak-type (p,q) if
$\|Tf\|_{L^{q,\infty}} \lesssim H\cdot W^{1/p}$ for all substep fcts $f\in D$ of height H
and width W.

a domain of T which is closed under addition, multiplic. by scalars and containing Σ_c.

The following result says that ~~of~~ restricted weak-type operators on characteristic functions are also of restricted weak-type on Σ_c.

<u>Proposition 1.3.5</u> Let $0<p\le\infty$, $0<q\le\infty$, $A>0$, and T be a sublinear operator defined on Σ_c. Then the following are equivalent.

(1) It holds that $\|Tf\|_{L^{q,\infty}} \lesssim A\, H\, W^{1/p}$ for all sub-step fcts f of height H, and width W

(2) For every set $F\subseteq Y$ of finite measure there exists a subset $F'\subseteq F$ with $\nu(F')\ge \frac{1}{2}\nu(F)$ and s.t. for all $E\subseteq X$ of finite measure,

$$\int_Y |(T\mathbb{1}_E)(y)|\, \mathbb{1}_{F'}(y)\, d\nu(y) \lesssim A\, \mu(E)^{1/p}\, \nu(F')^{1/q'}$$

<u>Proof</u> (1) $\Rightarrow$ (2) Follows from Thm 1.3.3

(2) $\Rightarrow$ (1) Assume first that f takes the form $f_N = \sum_{j=1}^N 2^{-j}\mathbb{1}_{E_j}$ where $\mu(E_j)\le W$

Then $\|Tf_N\|_{q,\infty} \lesssim A W^{1/p}$, uniformly in N, ~~as~~ will be shown.

First by the sublinearity of T and $\left\{\begin{array}{l}\|f+g\| \le c(\|f\|+\|g\|),\ \|f_j\| < 2^{-j}\cdot A \\ \Rightarrow \|\sum f_j\| \le A\cdot c_2 \end{array}\right.$ (exercise, Lemma 5.1 in Rey's notes)

It suffices to show $\|T\mathbb{1}_{E_j}\|_{q,\infty} \lesssim A W^{1/p}$ $(*)$.

Now let $F\subseteq Y$ be of finite measure. By assumption, $\exists\, F'\subseteq F$ with $\mu(F')\ge \frac{\mu(F)}{2}$ and $\int_Y \mathbb{1}_{F'}(y)\,(T\mathbb{1}_E)(y)\, d\nu(y) \lesssim A\,\mu(E)^{1/p}\,\nu(F')^{1/q'}$. But by Prop. 1.3.3 this is just what we asserted, i.e., $(*)$

So now let $0\le f\le 1$ be arbitrary function in Σ_c of width W, i.e.,

$$f = \sum_{j=1}^M a_j \mathbb{1}_{E_j} \quad \text{with pairwise disjoint } E_j \text{ (of finite measure)}$$

~~Now assume that $d_j(x)$~~

Denoting by $d_j(x)$ the j-th digit in the binary expansion of $f(x) = \sum_{j=1}^\infty d_j(x)\, 2^{-j}$

and defining $\underset{\substack{\wedge \\ Y\supseteq F_j \text{ through}}}{f_N} = \sum_{j=1}^N 2^{-j} d_j = \sum_{j=1}^N 2^{-j}\mathbb{1}_{F_j}$. Thus, $f - f_N = \sum_{j=1}^M b_{j,N}\mathbb{1}_{E_j}$ with $b_{j,N}\le 2^{-N}$

$$\Rightarrow \|Tf\|_{q,\infty} \lesssim \underbrace{\|Tf_N\|_{q,\infty}}_{\lesssim A W^{1/p}} + \underbrace{\|T(f-f_N)\|_{q,\infty}}_{\le \|\sum_{j=1}^M b_{j,N} |T\mathbb{1}_{E_j}|\|_{q,\infty} \lesssim 2^{-N}\|\sum_{j=1}^M |T\mathbb{1}_{E_j}|\|_{q,\infty}} \lesssim A W^{1/p}$$

$2^{-N}\|\sum_{j=1}^M |T\mathbb{1}_{E_j}|\|_{q,\infty}$ can be made arbitrarily small

By homogeneity & sublinearity we obtain the same bound (with a possibly larger constant) for arbitrary $f\in\Sigma_c$. $\square$

The proof of the Marcinkiewicz interpolation theorem relies on

Proposition 1.3.6 (Baby interpolation)

Let $0<p_0,p_1,q_0,q_1\le\infty$, $A_{0,1}>0$ and suppose T is a sublinear operator defined on Σ_c which is of restricted weak-type (p_i,q_i) with constants A_i $(i=0,1)$.

Then T is of restricted weak-type (p_θ,q_θ), i.e.,

$$\|Tf\|_{q_\theta,\infty}\lesssim_{p_0q_0p_1q_1} A_\theta \, \|f\|_{p_\theta} \quad \forall f\in\Sigma_c \quad \text{with } p_\theta, q_\theta, A_\theta \text{ as usual.}$$

Proof By Prop. 1.3.5 it suffices to show that for every $F\subseteq Y$ of finite measure, there exists $F'\subseteq F$ with $\mu(F')\geq\frac{\mu(F)}{2}$ s.t. for all $E\subseteq X$ of finite measure, one has

$$\int_Y \mathbb{1}_{F'}(y)(T\mathbb{1}_E)(y)\,d\nu(y)\lesssim_{p_0q_0p_1q_1} A_\theta\,\mu(E)^{1/p_\theta}\mu(F')^{1/q_\theta'}.$$

But by the assumptions for the end-points and Prop. 1.3.5 we already have this bound with the RHS replaced by $A_i\,\mu(E)^{1/p_i}\mu(F')^{1/q_i'}$ and so the result follows from scalar interpolation $X\le Y$ and $X\le Z \Rightarrow X\le Y^{1-\theta}Z^\theta$. ∎

We need one last lemma.

Lemma 1.3.7 Let $\Lambda_\lambda(x,y)=(1-\lambda)x+\lambda y$. Then $\Lambda_\alpha(\Lambda_\beta(x,y),\Lambda_\gamma(x,y))=\Lambda_{\Lambda_\alpha(\beta,\gamma)}(x,y)$

Proof Clear

Theorem 1.3.8 (Marcinkiewicz) Let T be a sublinear operator defined on Σ_c^+ (set of all non-neg fcts of finite supp) and $0<p_i,q_i\le\infty$ with $p_0\neq p_1$, $q_0\neq q_1$. If T satisfies $\|T\mathbb{1}_E\|_{q_i,\infty}\lesssim A_i\|\mathbb{1}_E\|_{L^{p_i,1}}$ $(i=0,1)$ for all $E\subseteq X$ of finite measure, then for all $1\le r\le\infty$ and $0<\theta<1$ s.t. $q_\theta>1$, we have $\|Tf\|_{L^{q_\theta,r}}\lesssim_{p_0p_1q_0q_1r\theta} A_\theta\|f\|_{L^{p_\theta,r}}$, $f\in\Sigma_c$

In particular, assuming $q_\theta\geq p_\theta$ taking $r=q_\theta$ (and using $L^{q,q}=L^q$, $L^{p,q}\supseteq L^p$), we see that T is of strong-type (p_θ,q_θ) with constant $C_{p_0p_1q_0q_1\theta}A_\theta$

Proof By Prop 1.3.5 we can assume that T is of restricted weak-type (p_i,q_i) with constants A_i. Let us first suppose $q_i\neq 1$. Let $f\in\Sigma^+$ which we may (by homogeneity) assume to be normalized s.t. $\|f\|_{p_\theta,r}=1$. By the dual characterization of L^{pq} (Thm 1.2.18) (here we need $\underline{q>1}$) it suffices to prove

$$\int_Y |(Tf)(y)|\, g(y)\, d\nu(y) \lesssim_{p_0 p_1 q_0 q_1 r \theta} A_\theta \ , \quad g \in \Sigma^+ \text{ with } \|g\|_{q_\theta', r'} \le 1$$

By assumption on T (restr. weak-type) + Hölder in $L^{p,q}$ we know

(*) $\displaystyle\int_Y |Tu|\, v\, d\nu \lesssim A_i\, H H' W^{1/p_i} W'^{1/q_i'} \lesssim A_\theta\, H H' \min_{i=0,1} \left(W^{1/p_i} W'^{1/q_i'}\right)$ for all substep fcts u, v of heights $\Sigma^+(X,\mu)$ $\Sigma^+(Y,\nu)$ H, resp H' and widths W, resp W'.

Now by the alternative characterization of $L^{p,q}$ (Thm 1.2.14), we can decompose

$L^{p_\theta, r} \ni f = \sum_m f_m$, f_m... substep fct of height H_m and width 2^m

$L^{q_\theta', r'} \ni g = \sum_n g_n$, g_n... substep fct of height H_n' and width 2^n

and the decompositions satisfy $\|H_m 2^{m/p_\theta}\|_{\ell^r_m} \sim \|H_n' 2^{n/q_\theta'}\|_{\ell^{r'}_n} \sim 1$. Moreover, recall that since f and g are simple, only finitely many H_m, H_n' are non-zero.

By (*) and sublinearity, we have

$$\int_Y |Tf| \cdot g\, d\nu \lesssim A_\theta \sum_{m,n} H_m H_n' \min_{i=0,1} \{2^{m/p_i}, 2^{n/q_i'}\}$$

Denote $a_m = H_m 2^{m/p_\theta}$ and $b_n = H_n' 2^{n/q_\theta'}$, then we need to show

(**) $\displaystyle\sum_{m,n} a_m b_n \min_{i=0,1} \{2^{m(\frac{1}{p_i} - \frac{1}{p_\theta})} \cdot 2^{n(\frac{1}{q_\theta'} - \frac{1}{q_i'})}\} \overset{!}{\lesssim} 1$, where $\|a_m\|_{\ell^r} = \|b_n\|_{\ell^{r'}} = 1$

Since $p_0 \neq p_1$ and $q_0 \neq q_1$ and because of the definitions of p_θ, q_θ, the LHS can be written as $\displaystyle\sum_{m,n} a_m b_n \min\{2^{\alpha\theta(m - \frac{\beta}{\alpha} n)}, 2^{-\alpha(1-\theta)(m - \frac{\beta}{\alpha} n)}\}$, $\alpha = \frac{1}{p_0} - \frac{1}{p_1}$ $\beta = \frac{1}{q_0} - \frac{1}{q_1}$

Now, write $\gamma = \beta/\alpha$ and translate $m \mapsto m + \lfloor \gamma n \rfloor$. Thus, we're left with showing

$$\underbrace{\sum_m \min\{2^{\alpha\theta m}, 2^{-\alpha(1-\theta)m}\} \sum_n a_{m+\lfloor\gamma n\rfloor} b_n}_{\le \underbrace{\|\min 2^{\alpha\theta m}, 2^{-\alpha(1-\theta)m}\|_{\ell^1_m}}_{\lesssim_{\alpha\theta} 1} \cdot \sup_m \underbrace{\left|\sum_n a_{m+\lfloor\gamma n\rfloor} b_n\right|}_{\le \underbrace{\|a_{m+\lfloor\gamma n\rfloor}\|_{\ell^r_n}}_{\lesssim_\gamma \|a^r_{m+n}\|_{\ell^1_n} \lesssim_{\gamma, r} 1} \underbrace{\|b_n\|_{\ell^{r'}_n}}_{=1 \text{ by assump}}} \lesssim_{\alpha,\theta,\gamma,r} 1} \le 1$$

(we used $2^{\alpha\theta m - \gamma n + \lfloor\gamma n\rfloor} \sim 2^{\alpha\theta m}$ and similarly for the 2nd factor)

All right! Now we only need to see how the assumption $q_i > 1$ can be dropped. (Exercise!)

↳ we used it to appeal to the dual characterization of $L^{p,q}$

~~Obser~~
To deal with $q \le 1$, observe that we can interpolate with the Baby Marcinkiewicz theorem to obtain that T has restricted weak-type for all $0 \le \theta \le 1$. Since we're imposing $q_\theta > 1$, we may assume $q_i > 1$ for at least one $i \in \{0,1\}$, (as the theorem is void otherwise). Thus, there must be $\tau \in (0,1)$ s.t. $1 < q_\tau < q_\theta$ and s.t. T is of restricted weak-type (p_τ, q_τ). But then we can interpolate ~~between~~ using what we have just proved from (p_τ, q_τ) to (p_i, q_i) (which has $q_i > 1$) and use Lemma 1.3.7 to ensure that (p_θ, q_θ) is in between (p_τ, q_τ) and (p_i, q_i).

1.3.2 Complex interpolation

Recall ~~the~~

Lemma 1.3.9 (Three-lines-lemma).
Let f be a complex-analytic (i.e., holomorphic) function on the strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ and continuous on the boundary. Suppose $|f(z)| \lesssim_f \exp(O_f(e^{(\pi - \delta)|z|}))$ in S for some $\delta > 0$

$$|f(z)| \le A, \quad \operatorname{Re} z = 0,$$
$$\text{and} \quad |f(z)| \le B \quad \operatorname{Re} z = 1 .$$

Then $|f(z)| \le A^{1-\operatorname{Re} z} \cdot B^{\operatorname{Re} z}$, $z \in S$.

Remark The (strange) sub-double exponential hypothesis here is completely sharp, as the example $\exp(-ie^{\pi i z})$ shows

Proof By homogeneity, we may assume $A = 1$, and $B = 1$ (since hypothesis and conclusion are invariant under multiplying f by a constant (to get rid of A), resp. under multiplying f by $\exp(cz)$ for some real c (to get rid of B)). Thus, f is bdd (in absolute value) by 1 on the borders of S by 1 and we want to show $|f(z)| \le 1$ in S.

Let us first assume that f behaves much better than exp. growth at infinity, namely that it decays to zero; then for all sufficiently large rectangles $\{0 < \operatorname{Re} z < 1;\ -N \le \operatorname{Im} z \le N\}$, the holomorphic fct f is bdd by 1 on all four sides of this rectangle, and hence, by the maximum modulus principle, by one also in the interior, namely by one. Thus, we're done, by letting $N \to \infty$

Now consider the general case. As is usual when removing a qualitative assumption, we do this by a limiting argument. We replace $f(z)$ by ~~$f(z)$~~ $f(z)\exp(\epsilon e^{i[(\pi-\epsilon)z+\epsilon/2]})$ which converts the almost double-exponentially growing function f to one which is still holomorphic but is now decaying at ∞. Moreover, it's still bdd by 1 at the borders of S and hence also by 1 in the interior by the previous argument. Taking $\epsilon \to 0$ yields the claim $\blacksquare$

Check that! (exercise!)

<u>Remark</u> The same reasoning can be used to show that, whenever $|f(z)| \lesssim (1+|z|)^{O(1)}$ on the borders of the strip, then the bound continues to hold in the interior.

<u>Theorem 1.3.10</u> (Riesz-Thorin interpolation)
Let T be a linear operator s.t. the form $\underbrace{\int g(x)(Tf)(x)\,d\mu(y)}_{(\langle g, Tf\rangle)}$ is well-defined $\forall f$.
Let $0 < p_0, p_1 \le \infty$, $1 \le q_0, q_1 \le \infty$ and $A_0, A_1 > 0$ be such that
$$\|Tf\|_{L^{q_i}(Y)} \le A_i \|f\|_{L^{p_i}(X)}, \quad f \text{ simple fct of finite measure support}$$
Then. $\|Tf\|_{L^{q_\theta}(Y)} \le A_\theta \|f\|_{L^{p_\theta}(X)}$, $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. $A_\theta = A_0^{1-\theta} A_1^{\theta}$, f simple fct of finite measure support.

<u>Proof</u> Wlog $(p_0, q_0) \neq (p_1, q_1)$. $A_0 = A_1 = 1$. By duality + homogeneity, it suffices to show
$$\left|\int g_\theta \, T f_\theta \, d\mu\right| \le 1, \quad \|f_\theta\|_{L^{p_\theta}(X)} = \|g_\theta\|_{L^{q_\theta'}(Y)} = 1 \quad \text{for all simple } f_\theta, g_\theta \text{ of finite measure support.}$$

The idea is to use the three-lines-lemma. However, the inequality is not holomorphic in θ as stated. We fix this as follows. Observe that if f_θ is simple with $\|f_\theta\|_{p_\theta} = 1$, we can factorize
$$f_\theta = \underbrace{\underset{L^{p_0}(X)}{\underset{\cap}{F_0}}^{1-\theta} \cdot \underset{L^{p_1}(X)}{\underset{\cap}{F_1}}^{\theta}}_{\text{non-negative, simple, normalized fcts}} \cdot a \quad \text{simple function with } |a(x)| \le 1.$$

Indeed, we can set $a = \operatorname{sgn} f$ and $F_i = |f_\theta|^{p_\theta/p_i}$ (Some minor changes need to be made for the limiting case when one or both of the p_i are equal to $\infty \to$ exercise).
Similarly, write $g_\theta = \underset{L^{q_0'}}{\underset{\cap}{G_0}}^{1-\theta} \cdot \underset{L^{q_1'}}{\underset{\cap}{G_1}}^{\theta} \cdot b$ (with similar meanings for G_0, G_1, b).

Now consider $H(z) := \int T(F_0^{1-z} F_1^{z} a)\, G_0^{1-z} G_1^{z}\, b \, d\nu$; since T is linear and all fcts are simple, it's easy to see that H is an entire fct of ~~z~~ of at most exp. growth; moreover it's bdd by 1 on the borders of S and hence bounded in S. Setting $z = \theta$ yields the claim $\blacksquare$

Comparison between Marcinkiewicz and Riesz-Thorin.

Advantages of R.-T. over M.: - we don't lose an unspecified constant
- we don't have the restriction $q_0 \geq p_0$.

Disadvantage: R.-T. requires strong-type rather than (restricted) weak-type control.

E. Stein made the fundamental observation that the R.-T. theorem can be easily enhanced ("by adding a single letter to the alphabet").

Theorem 1.3.11 (Stein interpolation)

Let T_z be a family of linear operators on $\{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$ s.t. for all $f \in D_X$, $g \in D_Y$, the form $\int_Y (T_z f)(y)\, g(y)\, d\nu(y)$ is absolutely convergent, holomorphic in S and continuous on the borders of S. Assume further that the form grows slower than double-exponentially in z.
Let $0 < p_0, p_1 \leq \infty$, $1 \leq q_0, q_1 \leq \infty$ and $A_0, A_1 > 0$ be such that $\|T_z f\|_{L^{q_i}(Y)} \leq A_i \|f\|_{L^{p_i}(X)}$ for all simple f of finite measure support, $i = 0, 1$ and $\operatorname{Re} z = i$ ($\operatorname{Re} z = 0$ or 1)
Then $\|T_\theta f\|_{L^{q_\theta}(Y)} \leq A_\theta \|f\|_{L^{p_\theta}(X)}$, f simple of finite measure support

Remarks (1) In fact, the boundedness assumptions on the borders can be weakened substantially. It suffices to require $\sup_{-\infty < y < \infty} e^{-b|y|} \log A_i(y) < \infty$
y is imaginary part of $z = \underbrace{\operatorname{Re} z}_{=0 \text{ or } 1} + i \cdot y$

~~where~~ for some $b < \pi$. See Stein-Weiss § V. 4, Thm 4.1 (p. 205)

(2) Riesz-Thorin is an immediate corollary by setting $T_z \equiv T$.

Proof We repeat the above argument. The only observation to make is that the function $H(z) := \int_Y T_z(F_0^{1-z} F_1^{z} a)\, G_0^{1-z} G_1^{z}\, b \, d\nu$ continues to be holomorphic.
This is easiest seen by decomposing all the simple functions into indicator functions.

Finally, we mention the following valuable observation of ~~Frank~~ R.L. Frank and J. Sabin.

Consider the following situation. Assume there is a linear operator T, initially defined on $C_c^\infty(\mathbb{R}^d)$ or $\mathcal{S}(\mathbb{R}^d)$. Suppose, we know that there is an extension, that we denote by T as well, that maps $L^p(\mathbb{R}^d)$ to $L^{p'}(\mathbb{R}^d)$ boundedly for $1<p<2<p'<\infty$. Then, by Hölder's inequality, we know that the operator $W_1 T W_2$ is $L^2(\mathbb{R}^d)$ bdd whenever $W_1, W_2 \in L^{2p/(2-p)}(\mathbb{R}^d)$.

Now suppose we obtained the $L^p \to L^{p'}$-bddness of T via Stein interpolation of some family T_z where $T_{iy}: L^2 \to L^2$ and $T_{-\lambda_0+iy}: L^1 \to L^\infty$ for some $\lambda_0>1$ i.e., we know that $T=T_{-1}$ is $L^p \to L^{p'}$ - bdd for $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{\lambda_0}$

$$-1 = (1-\theta)\cdot 0 - \theta \cdot \lambda_0 \Leftrightarrow \theta = 1/\lambda_0$$

$$\frac{1}{p} = \frac{1-1/\lambda_0}{2} + \frac{1}{\lambda_0}$$

Frank & Sabin observed that then, not only $W_1 T_{-1} W_2$ is L^2-bdd for $W_1, W_2 \in L^{2\lambda_0}$ but that $W_1 T_{-1} W_2 \in \mathfrak{S}^{2\lambda_0}(L^2(\mathbb{R}^d))$.

<u>Proposition 1.3.12</u> Let T_z be an analytic family of operators on $\mathbb{R}^d$ in the sense of Stein (i.e, of Thm 1.3.11) defined on the strip $\{z\in\mathbb{C}: -\lambda_0 \le \operatorname{Re} z \le 0\}$ for some $\lambda_0>1$.

Assume $\|T_{iy}\|_{L^2\to L^2} \le A_0 e^{a|y|}$ $\quad \|T_{-\lambda_0+iy}\|_{L^1\to L^\infty} \le A_1 e^{b|y|}$, $s\in\mathbb{R}$ for some $a,b \ge 0$ and $A_0, A_1 \ge 0$. Then for all $W_1, W_2 \in L^{2\lambda_0}(\mathbb{R}^d)$, the operator $W_1 T_{-1} W_2 \in \mathfrak{S}^{2\lambda_0}(L^2(\mathbb{R}^d))$ with $\|W_1 T_{-1} W_2\|_{\mathfrak{S}^{2\lambda_0}(L^2(\mathbb{R}^d))} \le A_0^{1-1/\lambda_0} A_1^{1/\lambda_0} \|W_1\|_{L^{2\lambda_0}} \|W_2\|_{L^{2\lambda_0}}$

<u>Proof</u> Exercise.