Winter term 2023/24,
to be discussed on
January 30, 2024

Harmonic Analysis
Homework Sheet 9

Exercise 9.1
Typically, the means of a sequence behave better than the original one. Recalling the Dirichlet kernel D_N, we consider the *Fejér kernel*

$$F_N(x) := \frac{1}{N+1} \sum_{j=0}^{N} D_j(x)\,, \quad x \in \mathbb{T}^1\,.$$

Show the following

Lemma 0.1. *For every $N \in \mathbb{N}$, we have the identities*

$$F_N(x) = \sum_{j=-N}^{N} \left(1 - \frac{|j|}{N+1}\right) \mathrm{e}^{2\pi i j x} = \frac{1}{N+1} \left(\frac{\sin(\pi(N+1)x)}{\sin(\pi x)}\right)^2\,, \quad x \in \mathbb{T}^1\,.$$

Thus,

$$\hat{F}_N(m) = \begin{cases} 1 - \frac{|m|}{N+1} & \text{if } |m| \leq N \\ 0 & \text{else} \end{cases}\,.$$

Correspondingly, one defines the *square Fejér kernel* $F_N^{(d)}$ on $\mathbb{T}^d$ as the product of the one-dimensional Fejér kernels, i.e., $F_N^{(d)}(x_1, ..., x_d) := \prod_{j=1}^{d} F_N(x_j)$.

Exercise 9.2
Show that the family of Fejér kernels $\{F_N^{(d)}\}_{N=0}^{\infty}$ form an approximate identity on $\mathbb{T}^d$; confront with Grafakos *Classical Fourier Analysis*, Definition 1.2.15, Theorem 1.2.19.
That is, show that $\int_{\mathbb{T}^d} F_N^{(d)}(x)\,dx = 1$, $\|F_N^{(d)}\|_{L^1(\mathbb{T}^d)} \lesssim 1$ uniformly in N, and, for given $\delta > 0$, one has $\int\limits_{|x|<1/2,\,|x|>\delta} F_N^{(d)}(x)\,dx \to 0$ as $N \to \infty$. (Consider first the case $d = 1$ and observe that the arguments easily generalize to $d \geq 2$.)

Exercise 9.3
We will now give a partial answer in which sense the partial sums of Fourier series may converge back to the original function when the cutoff is sent to infinity.
Let the *square Fejér mean* of a function f on $\mathbb{T}^d$ be defined as

$$(F_N^{(d)} * f)(x) = \sum_{m \in \mathbb{Z}^d,\, |m_j| \leq N} \prod_{j=1}^{d} \left(1 - \frac{|m_j|}{N+1}\right) \hat{f}(m_1, ..., m_d) \mathrm{e}^{2\pi i m_j x_j}\,.$$

Making use of this definition and the previous exercise, show the following

Lemma 0.2. *If $f, g \in L^1(\mathbb{T}^d)$ satisfy $\hat{f}(m) = \hat{g}(m)$ for all $m \in \mathbb{Z}^d$, then $f = g$ a.e.*

A useful conseuqence of this lemma is the following observation. Suppose $f \in L^1(\mathbb{T}^d)$ and $\sum_{m\in\mathbb{Z}^d} |\hat{f}(m)| < \infty$. Show that then $f(x) = \sum_{m\in\mathbb{Z}^d} \hat{f}(m)\mathrm{e}^{2\pi i m\cdot x}$ holds a.e. This shows that f is almost everywhere equal to a continuous function.

Exercise 9.4
We now partially connect Fourier analysis on $\mathbb{T}^d$ with Fourier analysis on $\mathbb{R}^d$ by showing that the Fourier series $\sum_{m\in\mathbb{Z}^d} \hat{f}(m)\mathrm{e}^{2\pi i m\cdot x}$ equals the periodization of f on $\mathbb{R}^d$. Show the following

Lemma 0.3 (Poisson summation)**.** *Let f be a continuous function on $\mathbb{R}^d$ such that for some δ, we have*

$$|f(x)| \lesssim (1 + |x|)^{-d-\delta}, \quad x \in \mathbb{R}^d .$$

Assume further that the Fourier transform $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\mathrm{e}^{-2\pi i x\cdot\xi}\,dx$ satisfies

$$\sum_{m\in\mathbb{Z}^d} |\hat{f}(m)| < \infty .$$

Then for all $x \in \mathbb{R}^d$, we have

$$\sum_{m\in\mathbb{Z}^d} \hat{f}(m)\mathrm{e}^{2\pi i m\cdot x} = \sum_{k\in\mathbb{Z}^d} f(x+k) ,$$

and in particular

$$\sum_{m\in\mathbb{Z}^d} \hat{f}(m) = \sum_{k\in\mathbb{Z}^d} f(k) .$$

(Hint: Define the 1-periodic function $F(x) = \sum_{k\in\mathbb{Z}^d} f(x+k)$ on $\mathbb{T}^d$ and prove $\hat{F}(m) = \hat{f}(m)$. This allows you to use the preceding exercise.)