Winter term 2023/24,
to be discussed on
January 23, 2024

Harmonic Analysis
Homework Sheet 8

Exercise 8.1
Compute the Dirichlet kernel, i.e., for $t \in [0,1]$ and $N \in \mathbb{N}$ show that

$$D_N(t) := \sum_{m=-N}^{N} \mathrm{e}^{2\pi i m t} = \frac{\sin((2N+1)\pi t)}{\sin(\pi t)}\,.$$

Note that the higher-dimensional *square Dirichlet kernel*, i.e.,

$$D_N^{\square}(t) := \sum_{m \in \mathbb{Z}^d,\, |m_j| \le N} \mathrm{e}^{2\pi i m\cdot t} = \prod_{j=1}^{d} D_N(t_j)\,, \quad t \in \mathbb{T}^d$$

simply factorizes (as opposed to the *spherical Dirichlet kernel* $\sum_{m\in\mathbb{Z}^d,\,|m|\le N} \mathrm{e}^{2\pi i m\cdot t}$).

Exercise 8.2
The disc conjecture for $1 < p < \infty$ is the statement that the Fourier multiplier $\mathbf{1}_{B_0(1)}(\xi)$ is bounded on $L^p(\mathbb{R}^2)$, i.e., $\|(\mathbf{1}_{B_0(1)}\hat{f})^\vee\|_p \lesssim_p \|f\|_p$. (The case $p=2$ is trivial in view of Plancherel's theorem.) Show the following

Lemma 0.1 (Y. Meyer)**.** *Let $(v_j)_{j\in\mathbb{N}} \in \mathbb{S}^1$ be a sequence of unit vectors in $\mathbb{R}^2$ and let H_j be the half-plane $\{x \in \mathbb{R}^2 : x \cdot v_j \ge 0\}$. Define the "half plane multipliers" $(T_j)_{j\in\mathbb{N}}$ on $L^p(\mathbb{R}^2)$ by setting $\widehat{T_j f}(\xi) = \mathbf{1}_{H_j}(\xi)\hat{f}(\xi)$. If the disc conjecture holds for some $1 < p < \infty$, then for any sequence $(f_j)_{j\in\mathbb{N}}$, we have the square function estimate*

$$\|(\sum_j |T_j f_j|^2)^{1/2}\|_p \lesssim_p \|(\sum_j |f_j|^2)^{1/2}\|_p\,, \quad f_j \in \mathcal{S}(\mathbb{R}^2)\,.$$

Exercise 8.3
Truncated integral kernels usually give rise to somewhat "localized" operators. For $R > 0$, we say that a linear operator T is *R-local* if $\operatorname{supp} Tf \subseteq R \operatorname{supp} f$. (Here, $RA = \{x \in \mathbb{R}^d : x \in B_y(R) \text{ for any } y \in A\}$ for some $A \subseteq \mathbb{R}^d$.) In particular, this means that $\operatorname{supp} T\mathbf{1}_{B_x(R)} \subseteq B_x(2R)$. Show the following

Lemma 0.2. *Suppose T is a R-local linear operator taking functions on $\mathbb{R}^d$ to functions on $\mathbb{R}^d$. Then, for any $1 \le p \le q \le \infty$, the bound*

$$\|Tf\|_q \lesssim \|f\|_p\,, \quad f \in L^p(\mathbb{R}^d) \tag{1}$$

is equivalent to the bound

$$\|Tf\|_{L^q(B_x(2R))} \lesssim \|f\|_p\,, \quad f \in L^p(B_x(R)) \tag{2}$$

holding uniformly in $x \in \mathbb{R}^d$. In other words, to show (1) it suffices to test it for functions on an R-ball.