Winter term 2023/24,
to be discussed on
January 16, 2024

Harmonic Analysis
Homework Sheet 7

Exercise 7.1
Recall the Dirichlet kernel

$$D_N(t) = \frac{\sin((2N+1)\pi t)}{\sin(\pi t)}$$

for $N \in \mathbb{N}$ and $t \in [0,1]$. Prove $|1/t - 1/(\sin t)| \lesssim 1$ (by using, e.g., the fundamental theorem of calculus) for $|t| \leq \pi/2$ to show $\|D_N\|_{L^1([0,1])} \sim \log N$. (In fact, one can prove $|1/t - 1/(\sin t)| \leq 1 - 2/\pi$ and

$$\frac{4}{\pi^2}\sum_{k=1}^{N}\frac{1}{k} \leq \|D_N\|_1 \leq 3 - \frac{2}{\pi} + \frac{4}{\pi^2}\sum_{k=1}^{N}\frac{1}{k}\,,$$

but this is not required here.)

Exercise 7.2
Prove the following

Lemma 0.1 (Weighted Schur test)**.** *Let (X,μ) and (Y,ν) be measure spaces and $w(x,y) > 0$ be measurable on $X \times Y$. Suppose the kernel $K(x,y) : X \times Y \to \mathbb{C}$ satisfies*

$$\sup_{x\in X}\int_Y w(x,y)^{1/p}|K(x,y)|\,d\nu(y) \equiv A_1 < \infty \quad \text{and} \quad \sup_{y\in Y}\int_X w(x,y)^{-1/p'}|K(x,y)|\,d\mu(x) \equiv A_2 < \infty$$

for some $1 < p < \infty$, respectively $1 \leq p \leq \infty$ when $w(x,y) \equiv 1$. Then, the operator T defined by $(Tf)(x) = \int_Y K(x,y)f(y)\,d\nu(y)$ satisfies $\|Tf\|_{L^p(X)} \leq A_1^{1/p'}A_2^{1/p}\|f\|_{L^p(Y)}$.

Exercise 7.3
Show the following transference-type

Lemma 0.2 (de Leeuw)**.** *Suppose that m is a smooth Fourier multiplier on $\mathbb{R}^d$ and that the operator T defined by*

$$\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$$

is bounded on $L^p(\mathbb{R}^d)$. Then the operator T_0 defined by

$$\widehat{T_0 g}(\xi') = m(\xi',0)\hat{g}(\xi')$$

for $\xi' \in \mathbb{R}^{d-1}$ is bounded on $L^p(\mathbb{R}^{d-1})$.