Winter term 2023/24,
to be discussed on
December 12, 2023

Harmonic Analysis
Homework Sheet 5

Exercise 5.1
Establish b) and d) in Theorem 2.6 in the notes.

Exercise 5.2
Compute $H(\mathbf{1}_{[a,b]})(x)$ for $-\infty < a < b < \infty$.

Exercise 5.3
We establish another proof of $\widehat{H(\varphi)} = -i\,\mathrm{sgn}(\xi)\hat{\varphi}$ and another equivalent definition of the Hilbert transform. Let

$$Q_t = \frac{1}{\pi}\frac{x}{x^2+t^2}$$

be the conjugate Poisson kernel of the last sheet. Show that $\lim_{t\to 0} Q_t = \pi^{-1}\,\mathrm{p.\,v.}(x^{-1})$ in $\mathcal{S}'$, i.e.,

$$\lim_{t\to 0}\langle Q_t, \varphi\rangle_{\mathcal{S}} = H(\varphi)\,, \quad \varphi \in \mathcal{S}(\mathbb{R})\,.$$

(We will later on show L^p and pointwise a.e. convergence (by considering the corresponding maximal function) for $\varphi \in L^p$ with $p \in [1,\infty)$, thereby extending the Hilbert transform from $\mathcal{S}$.) By the continuity of the Fourier transform in $\mathcal{S}'$ and $\hat{Q}_t(\xi) = -i\,\mathrm{sgn}(\xi)\mathrm{e}^{-2\pi t|\xi|}$ (from the last sheet), this shows once more

$$\widehat{H(\varphi)}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{\varphi}(\xi)$$

which, in turn, allows us to define H in $L^2(\mathbb{R})$ by Plancherel's theorem. Show that

$$\|Hf\|_2 = \|f\|_2, \tag{1}$$
$$H(Hf) = -f, \tag{2}$$
$$H^* = -H. \tag{3}$$

Exercise 5.4

1. Let $0 < a < b < \infty$. Show that $|\int_a^b x^{-1}\sin x\,dx| \leq 4$.
2. Consider the one-dimensional Laplace transform of $\sin(x)/x$, i.e.,

   $$I(a) := \int_0^\infty \mathrm{e}^{-ax}\frac{\sin x}{x}\,dx\,, \quad a > 0\,.$$

   Show that $I(a)$ is continuous at $a = 0$ and differentiate with respect to a to conclude $I(a) = \pi/2 - \arctan(a)$ and $I(0) = \pi/2$. ($I(0)$ is known as the Dirichlet integral.) Finally, deduce

   $$\int_{\mathbb{R}} \frac{\sin(x\xi)}{x}\,dx = \pi\,\mathrm{sgn}(\xi)\,.$$

   (Hint: Integration by parts and $\partial_x(1-\cos x) = \sin x$ may be helpful.)