Winter term 2023/24,
to be discussed on
November 28, 2023

Harmonic Analysis
Homework Sheet 4

Exercise 4.1
Show that the Hardy–Littlewood maximal operator M is never $L^1(\mathbb{R}^d)$-bounded and that it is local in the sense that if $Mf(x_0) = 0$ for some $x_0 \in \mathbb{R}^d$, then $f = 0$ a.e. (Hint: Consider $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and average it over a fixed ball $B_x(|x| + R)$.)

Exercise 4.2
Consider the *uncentered* Hardy–Littlewood maximal function of f,

$$(\mathcal{M}f)(x) := \sup_{R>0,\, |x-y|\le R} \frac{1}{|B_y(R)|} \int_{B_y(R)} |f(z)|\, dz$$

which is the supremum over all averages of $|f|$ over all open balls $B_y(R)$ containing the point $x \in \mathbb{R}^d$. Show that $Mf \le \mathcal{M}f \le 2^d Mf$ pointwise. (This tells us that $\mathcal{M}$ inherits all boundedness properties of M and vice versa.)

Exercise 4.3
Establish Theorem 2.1 (the "analog" of the uniform boundedness principle) in the notes and use it to prove Lebesgue's differentiation theorem

$$\lim_{r\to 0} \frac{1}{|B_x(r)|} \int_{B_x(r)} f(y)\, dy = f(x)$$

for a.e. $x \in \mathbb{R}^d$, whenever $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

Exercise 4.4
Let $t > 0$ and compute the d-dimensional Poisson kernel

$$\int_{\mathbb{R}^d} \mathrm{e}^{-2\pi t|\xi| - 2\pi i x\cdot\xi}\, d\xi = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}} \cdot \frac{t}{(t^2 + |x|^2)^{(d+1)/2}}\,, \quad x \in \mathbb{R}^d$$

(with $\Gamma(z) = \int_0^\infty \mathrm{e}^{-s} s^{z-1}\, ds$ for $\mathrm{Re}\, z > 0$) and the one-dimensional conjugate Poisson kernel

$$\int_{\mathbb{R}} \mathrm{sgn}(\xi) \mathrm{e}^{-2\pi t|\xi| + 2\pi i x\cdot\xi}\, d\xi = -\frac{i}{\pi} \cdot \frac{x}{x^2 + t^2}\,, \quad x \in \mathbb{R}\,.$$

(Hint: To compute the Poisson kernel, you may use (and prove) the identities

$$\int_{\mathbb{R}^d} \mathrm{e}^{-\pi t|\xi|^2 - 2\pi i x\cdot\xi}\, d\xi = t^{-d/2} \mathrm{e}^{-\pi|x|^2/t}$$

and

$$\mathrm{e}^{-\beta} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\mathrm{e}^{-u}}{\sqrt{u}} \mathrm{e}^{-\beta^2/(4u)}\, du\,, \quad \beta > 0\,.)$$