Winter term 2023/24,
to be discussed on
November 21, 2023

Harmonic Analysis
Homework Sheet 3

Exercise 3.1

Establish Theorem 1.3.4 in the case where one of the $q_i \le 1$. The following lemma might be helpful.

Lemma 0.1. *Let* $\Lambda_\lambda(x,y) = (1-\lambda)x + \lambda y$ *for* $0 \le \lambda \le 1$ *and* $x, y > 0$. *Then,*

$$\Lambda_\alpha(\Lambda_\beta(x,y), \Lambda_\gamma(x,y)) = \Lambda_{\Lambda_\alpha(\beta,\gamma)}(x,y)\,.$$

Exercise 3.2

Let $A, B_0, B_1 \ge 0$ such that $A \le \min(B_0, B_1)$. Show that for all $\theta \in (0,1)$ we have

$$A \le B_\theta \min(\frac{B_0}{B_1}, \frac{B_1}{B_0})^\varepsilon$$

for any sufficiently small $\varepsilon_\theta > 0$, where $B_\theta = B_0^{1-\theta} B_1^\theta$. Can you imagine a situation where this type of estimate is superior to the simpler estimate $A \le B_\theta$?

Exercise 3.3

In Lemma 1.3.5, show that the sub-double exponential hypothesis $|f(z)| \lesssim_f \exp(\mathcal{O}_f(\mathrm{e}^{(\pi-\delta)|z|}))$ for some $\delta > 0$ is completely sharp.

Exercise 3.4

Prove Hadamard's three circle theorem. Let $g(z)$ be holomorphic on the annulus $A := \{z \in \mathbb{C} : r_1 \le |z| \le r_3\}$ for some $0 < r_1 < r_3$ and denote

$$M(r) := \max_{\theta \in [0,2\pi]} |g(r\mathrm{e}^{i\theta})| \quad \text{for } r \in (r_1, r_3)\,.$$

Prove that

$$\log(\frac{r_3}{r_1}) \log M(r_2) \le \log(\frac{r_3}{r_2}) \log M(r_1) + \log(\frac{r_2}{r_1}) \log M(r_3)$$

for any $r_1 < r_2 < r_3$, i.e., $\log M(r)$ is a convex function of $\log r$. (Hint: Convince yourself that there is a vertical strip which the exponential function maps onto the annulus A.)

Compute both sides of the inequality when $g(z) = cz^\lambda$ for some constants $c \in \mathbb{C}$ and $\lambda \in \mathbb{Z}$.