Winter term 2023/24,
to be discussed on
November 14, 2023

# Harmonic Analysis
# Homework Sheet 2

**Exercise 2.1**
Let $\hat{f}(\xi) := \int_{\mathbb{R}^d} \mathrm{e}^{2\pi i x\cdot\xi} f(x)\,dx$ denote the Fourier transform which is well-defined on $L^1(\mathbb{R}^d)$ or the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and then extended to $L^p(\mathbb{R}^d)$ for any $1 \le p \le 2$ by Plancherel (initially on $L^1 \cap L^2$ and then extended via density to $L^2$) and interpolation. Recall that the interpolation lead us to the (non-optimal) Hausdorff–Young inequality

$$\|\hat{f}\|_{L^{p'}(\mathbb{R}^d)} \le \|f\|_{L^p(\mathbb{R}^d)}\,.$$

Suppose there was an inequality of the form

$$\|\hat{f}\|_{L^q(\mathbb{R}^d)} \le C_{p,q,d}\|f\|_{L^p(\mathbb{R}^d)}$$

for some $1 \le p, q \le \infty$. Show (by a scaling argument) that necessarily $q = p'$ and, by randomizing a sequence of functions and Khintchine's inequality, that $p \le 2$.

**Exercise 2.2**
Let $f, g$ be measurable on a $\sigma$-finite measure space $(X, \mu)$. Prove the Hardy–Littlewood inequality

$$\int_X |f(x)g(x)|\,d\mu(x) \le \int_0^\infty f^*(t)g^*(t)\,dt,$$

where $f^*, g^*$ are the decreasing rearrangements of $f$ and $g$, respectively.

**Exercise 2.3**
Show the following

**Lemma 0.1.** *Let $(X, \|\cdot\|)$ be a quasi-normed space, i.e., $\|f+g\| \le c_1(\|f\| + \|g\|)$ for some $c_1 \ge 1$. Assume that a sequence $(f_k)_{k\in\mathbb{N}} \in X$ satisfies $\|f_k\| \lesssim A\cdot c_2^{-k}$ for some $A > 0$ and $c_2 > 1$. Then $\|\sum_{k=1}^N f_k\| \le A\cdot c_3$ where $c_3$ does not depend on $A$ or $N$ (but possibly on $c_1$ and $c_2$).*

(Why is this assertion non-trivial?)

**Exercise 2.4 (optional)**
Establish the upper bound "$\lesssim_{p,q}$" in Proposition 1.2.18 for $q < \infty$. (Instruction/Hints: Make use of (i)⇒(ii) of Theorem 1.2.14 to decompose $f = \sum_m f_m$ where $f_m$ are quasi-step functions of height $2^m$ and width $W_m$ so that the sequence $a_m := 2^m W_m^{1/p}$ has $\ell^q_m$ norm $\sim_{p,q} 1$. Then make an ansatz for $g$ such as $g := \sum_m g_m$ where $g_m := a_m^r |f_m|^{p-2}\overline{f_m}$ with $r = q - p$ (or $r = (q-p)_+$) and show that $|\int_X fg| \sim_{p,q} 1$. Thus, you have reduced the claim to showing $\|\sum_m g_m\|_{p',q'} \lesssim_{p,q} 1$ which you may want to show using (iii)⇒(i) of Theorem 1.2.14. First of all, convince yourself that $g_m \lesssim_{p,q} a_m^r 2^{m(p-1)}\mathbf{1}_{E_m}$ where $E_m = \operatorname{supp}\ f_m$ satisfies $\mu(E_m) \lesssim_{p,q} W_m = 2^{-mp}a_m^p$ with the

above $a_m$. To remedy for the fact that the sequence of heights of $g_m$ is not lacunary (at least a priori), you can introduce modified heights $H_m := \sup_{k\geq 0} a_{m-k}^r 2^{m(p-1)} 2^{-k(p-1)/2}$. (Clearly, the old weights are recovered for $k=0$.) Having checked that $H_{m+1} \geq 2^{(p-1)/2} H_m$ (i.e., they increase geometrically), we see that it would suffice to check $\|\sum_m H_m \mathbf{1}_{E_m}\|_{p',q'} \lesssim_{p,q} 1$. But now, we're in position to apply (iii)⇒(i), i.e., you are left to check $\|H_m \mu(E_m)^{1/p'}\|_{\ell_m^{q'}} \lesssim_{p,q} 1$.)

**Exercise 2.5 (optional)**
Let $0 < p \leq \infty$, $1 < q \leq \infty$ and assume $(X,\mu)$ and $(Y,\nu)$ are two measure spaces. Let $T$ be a sublinear operator (initially defined on the set of really simple funcions $f = \sum_{k=1}^N a_k \mathbf{1}_{E_k}$ on $X$ such that $Tf$ is a $\nu$-measurable function on $Y$), i.e., $|T(f+g)| \leq |Tf| + |Tg|$ and $|T(\lambda f)| = |\lambda||Tf|$ for $f,g \in \mathrm{dom}(T)$ and $\lambda \in \mathbb{C}$. We say that $T$ is of restricted weak type $(p,q)$ if

$$\alpha d_{T\mathbf{1}_E}(\alpha)^{1/q} \lesssim |E|^{1/p} \quad \text{for all } \alpha > 0\,, E \subseteq X\,.$$

Prove that $T$ is of restricted weak type $(p,q)$ if and only if

$$|\int_F (T\mathbf{1}_E)(x)\, d\nu(x)| =: |\langle \mathbf{1}_F, T\mathbf{1}_E\rangle| \lesssim |E|^{1/p} |F|^{1/q'}$$

for all $E \subseteq X$ and $F \subseteq Y$. (Hint: Use Proposition 1.2.18 for "⇐" and Hölder's inequality or the layer-cake representation and Fubini to prove "⇒".)