Harmonic Analysis
Homework Sheet 11

Exercise 11.1
Assume $A : \mathcal{D}(A) \to \mathcal{H}$ is a linear non-negative (i.e., in particular self-adjoint) operator in some Hilbert space $\mathcal{H}$. Suppose, we knew that a Hörmander spectral multiplier theorem held for this operator such as the following

Theorem 0.1. *Let $\sigma > 0$, fix $0 \not\equiv \omega \in C_c^\infty(\mathbb{R}_+)$, and suppose F is a bounded and measurable function on $\mathbb{R}$ such that*

$$\sup_{t>0} \|\omega(\cdot)F(t\cdot)\|_{H^\sigma(\mathbb{R})} < \infty \tag{1}$$

Then $F(A)$ is $L^p(\mathbb{R}^d)$ bounded for all $1 < p < \infty$.

Now, let $\Phi : [0, \infty) \to [0, 1]$ be a smooth, compactly supported function such that

$$\Phi(\lambda) = 1 \text{ for } 0 \le \lambda \le 1 \quad \text{and} \quad \Phi(\lambda) = 0 \text{ for } \lambda \ge 2.$$

For a dyadic number $N \in 2^{\mathbb{Z}}$, we define

$$\Phi_N(\lambda) = \Phi(\lambda/N) \quad \text{and} \quad \Psi_N(\lambda) = \Phi_N(\lambda) - \Phi_{N/2}(\lambda) \in C_c^\infty(\mathbb{R}_+).$$

We see that $\{\Psi_N(\lambda)\}_{N\in 2^{\mathbb{Z}}}$ constitutes a partition of unity for $\lambda \in \mathbb{R}_+$. Using these functions, we define the standard Littlewood–Paley projections (via the L^2 functional calculus) as

$$P_N := \Psi_N(\sqrt{A})\,.$$

Using these projections, prove

Theorem 0.2 (Square function estimates)**.** *Let $s > 0$ and $1 < p < \infty$. Assume $k \in \mathbb{N}$ such that $2k > s$. Then we have*

$$\|A^{\frac{s}{2}} f\|_p \sim \left\| \left(\sum_{N\in 2^{\mathbb{Z}}} \left| N^{s/2} (P_N)^k f \right|^2 \right)^{\frac{1}{2}} \right\|_p$$

for all $f \in C_c^\infty(\mathbb{R}^d)$.

Exercise 11.2
Suppose $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is such that $-\Delta + V$ can be realized as a linear, densely defined operator on $\mathcal{D}(-\Delta) \cap \mathcal{D}(V)$. *Formally* establish the following Duhamel formula,

$$\mathrm{e}^{\Delta} - \mathrm{e}^{-(-\Delta+V)} = \int_0^1 \mathrm{e}^{(1-s)\Delta} V \mathrm{e}^{-s(-\Delta-V)}\, ds\,.$$

(Hint: The fundamental theorem of calculus may be helpful.)

Exercise 11.3
Using the three-lines lemma, establish the following Phragmén–Lindelöf

Lemma 0.3. *Let F be holomorphic on $\mathbb{C}_+ := \{z \in \mathbb{C} : \Re(z) > 0\}$ and assume the bounds*

$$|F(r\mathrm{e}^{i\theta})| \leq a_1 (r\cos\theta)^{-\beta}$$
$$|F(r)| \leq a_1 r^{-\beta} \exp(-a_2 r^{-\alpha})$$

for some $a_1, a_2 > 0$, $\beta \geq 0$, $0 < \alpha \leq 2$, all $r > 0$, and $|\theta| < \pi/2$. Then, the estimate

$$|F(r\mathrm{e}^{i\theta})| \leq a_1 2^{\beta} (r\cos\theta)^{-\beta} \exp\left(-\frac{1}{2} a_2 \alpha r^{-\alpha} \cos\theta\right)$$

holds for all $r > 0$ and $|\theta| < \pi/2$.