Winter term 2023/24,
to be discussed on
February 06, 2024

Harmonic Analysis
Homework Sheet 10

Exercise 10.1
Let $m \in C^\infty(\mathbb{R}^d)$ be homogeneous of imaginary order, i.e., for some fixed $\tau \in \mathbb{R}$ and all $\lambda > 0$, one has $m(\lambda\xi) = \lambda^{i\tau} m(\xi)$. Show that then

$$|\partial_\xi^\alpha m(\xi)| \lesssim_\alpha |\xi|^{-|\alpha|} \quad \text{for all } \alpha \in \mathbb{N}_0^d\,. \tag{1}$$

Let $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geq 0$. Show that for $\xi = (\xi_1, ..., \xi_d)$ the functions

$$m_0(\xi_1,\xi_2,\xi_3) = \frac{\xi_1^2+\xi_2^2}{\xi_1^2 + i(\xi_2^2+\xi_3^2)}\,, \quad m_1(\xi) = \left(\frac{|\xi|^2}{1+|\xi|^2}\right)^z, \quad \text{and} \quad m_2(\xi) = \left(\frac{1}{1+|\xi|^2}\right)^z$$

defined on $\mathbb{R}^d$ satisfy (1). (As we will see later, this shows that the m_j give rise to L^p bounded Fourier multipliers by the Mikhlin–Hörmander theorem.)

Exercise 10.2
Let $\delta \geq 0$ and $f \in \mathcal{S}(\mathbb{R}^d)$. Show that

$$(S_1^\delta f)(x) := \int_{\mathbb{R}^d} (1-|\xi|^2)_+^\delta \mathrm{e}^{2\pi i x\cdot\xi} \hat{f}(\xi)\, d\xi \sim \int_{\mathbb{R}^d} \frac{\sum_\pm \mathrm{e}^{\pm 2\pi i |x-y|} + o_{|x-y|\to\infty}(1)}{1+|x-y|^{(d+1)/2+\delta}} f(y)\, dy$$

(Hints: You may use the following facts. For a radial function $f \in \mathcal{S}(\mathbb{R}^d)$ with $d \geq 2$, its Fourier transform is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) \mathrm{e}^{-2\pi i x\cdot\xi}\, dx = 2\pi|\xi|^{-(d-1)} \int_0^\infty f(r) J_{(d-2)/2}(2\pi r|\xi|)(|r||\xi|)^{d/2}\, dr$$

where $J_\nu(x)$ ($\nu \in \mathbb{R}$) is the Bessel function of the first kind. For $\mu > -1/2$, the recursion relation

$$J_{\mu+\nu+1}(t) = \frac{t^{\nu+1}}{2^\nu \Gamma(\nu+1)} \int_0^1 J_\mu(ts) s^{\mu+1}(1-s^2)^\nu\, ds\,.$$

holds, whenever $\nu > -1$ and $t > 0$. Finally, for $\nu \in \mathbb{R}$, the asymptotic expansion

$$J_\nu(r) = \sqrt{\frac{2}{\pi r}} \cos(r - \pi\nu/2 - \pi/4) + \mathcal{O}(r^{-3/2}) \quad \text{as } r \to \infty$$

holds. These facts are proved in Chapter IV, Theorem 3.3, Lemma 4.13, and Lemma 3.11 in Stein–Weiss, respectively.) Use Schur's test to conlude that S_1^δ defines an L^p bounded Fourier multiplier for all $1 \leq p \leq \infty$ whenever $\delta > (d-1)/2$. (Note that the integral kernel of S_1^δ is analytic, i.e., there is no singularity at zero, since $(1-|\xi|^2)_+^\delta$ is compactly supported.)

Exercise 10.3
Let S_1^δ be the above Fourier multiplier. Show that $\|S_1^\delta f\|_{L^p(\mathbb{R}^d)} \lesssim_{p,\delta} \|f\|_{L^p(\mathbb{R}^d)}$ necessarily implies

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{2\delta+1}{2d}\,.$$

What does this result tell us when δ increases? Taking into account the preceding exercise, comment on the case $d = 1$ (in view of convergence of Fourier series).