SUMMARY OF IONESCU–SCHLAG AND SOME OTHER THINGS

1. Summary of Ionescu–Schlag [7]

Recall some notation.

- $B = \{f : \mathbb{R}^d \to \mathbb{C} : \|f\|_B^2 := \sum_{j\ge 0} 2^j \int_{|x|\in[2^{j-1},2^j]} |f|^2 < \infty\}$ and $B^* = \{g : \mathbb{R}^d \to \mathbb{C} : \|g\|_{B^*}^2 := \sup_{j\ge 0} 2^{-j} \int_{|x|\in[2^{j-1},2^j]} |g|^2 < \infty\}$ the classical Agmon–Hörmander spaces. Note $B \hookrightarrow L^2 \hookrightarrow B^*$.
- $S_\alpha : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ with $S_\alpha f = (1-\Delta)^{\alpha/2} \equiv < \nabla >^\alpha f$ and $\alpha \in \mathbb{C}$. (For $\alpha \in \mathbb{R}$ this corresponds to fractional differentiation/integration).
- $W^{\alpha,p} = \{u \in \mathcal{S}'(\mathbb{R}^d) : S_\alpha u \in L^p\}$ with $\|u\|_{W^{\alpha,p}} = \|S_\alpha u\|_{L^p}$ denote the usual Sobolev spaces. If $p = 2$, then $W^{\alpha,2} \equiv H^\alpha$.
- For $p_d = (2d+2)/(d+3)$ the main Banach spaces[1] here are

irreg. + schnell abf.

$$X = W^{-1/(d+1),p_d} + S_1(B)\,, \quad \|f\|_X = \inf_{f=f_1+f_2} \|S_{-1/(d+1)} f_1\|_{L^{p_d}} + \|S_{-1} f_2\|_B$$

reg. + langsam abfallend

$$X^* = W^{1/(d+1),p_d'} \cap S_{-1}(B^*)\,, \quad \|g\|_{X^*} = \max\{\|S_{1/(d+1)} g\|_{L^{p_d'}}, \|S_1 g\|_{B^*}\}$$

Note that $X^* \subseteq H^1_{\mathrm{loc}}$ and $X \hookrightarrow H^{-1}$ and $H^1 \hookrightarrow X^*$.

$R_0(z) = (-\Delta - z)^{-1}$

Theorem 1.1 (Combined LAP). *Let $\delta \in (0,1]$, then*

ℓ?

$$\sup_{|\lambda|\in(\delta,\delta^{-1})} \sup_{\varepsilon\in(-1,1)\setminus 0} \|R_0(\lambda \pm i\varepsilon)\|_{X\to X^*} \lesssim_\delta 1\,.$$

The following is a weighted version thereof and will be used to establish[2]

$\ell^p \hookrightarrow \ell^q$
$p<q$

- discreteness of point spectrum in $\mathbb{R}\setminus 0$ and
- rapid decay of eigenfunctions.

For $N \ge 0$ and $\gamma \in (0,1]$, let

$$\mu_{N,\gamma}(x) = \frac{< x >^{2N}}{< \sqrt{\gamma} x >^{2N}}$$

$\gamma \to 0$ $\quad <x>^{2n}$

which equals $< x >^{2N}$ for $\gamma \to 0$ and 1 for $\gamma = 1$.

Theorem 1.2 (Weighted combined LAP). *Let $\delta \in (0,1]$, then*

$$\|\mu_{N,\gamma} u\|_{X^*} \lesssim_{N,\delta} \|\mu_{N,\gamma}(-\Delta - \lambda) u\|_X\,, \quad |\lambda| \in (\delta, \delta^{-1})$$

whenever $u \in X^*$ satisfies the mild decay condition (cf. Lemma 4.3)

$$\lim_{R\to\infty} R^{-1} \int_{R\le|x|\le 2R} |u|^2 = 0\,.$$

[1] X: fast decay, low regularity — X^*: slow decay, good regularity.

[2] Compare with [3, Theorem 14.2.4].

$H^1 \hookrightarrow X^* \subset H^1_{loc}$

Definition (Admissible potentials). *Let $\mathfrak{B}(X^*, X)$ denote the space of linear bounded operators from X^* to X. Then V is said to be* admissible *if*

(1) *$V \in \mathfrak{B}(X^*, X)$ and $(\psi, V\varphi) = \overline{(\varphi, V\psi)}$ for any $\psi, \varphi \in \mathcal{S}(\mathbb{R}^d)$.*

(2) *For all $\varepsilon > 0$ and $N \geq 0$ there exist $A_{N,\varepsilon}, R_{N,\varepsilon} \geq 1$ such that*

$V: X^* \to X$

$$\|\mu_{N,\gamma} V u\|_X \leq \varepsilon \|\mu_{N,\gamma} u\|_{X^*} + A_{N,\varepsilon} \|u \mathbf{1}_{|x| \leq R_{N,\varepsilon}}\|_{L^2}\,, \quad u \in X^*\,, \gamma \in (0,1]\,.$$

$\langle x \rangle^N$

(Observe what happens if u solves $-\Delta + Vu = \lambda u$ in view of Theorem 1.2).

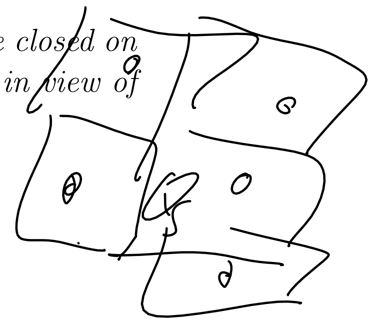
(3) *There exist $J \in \mathbb{N}_{>0}$ and operators $A_j, B_j \in \mathfrak{B}(X^*, L^2)$ $(j = 1, 2, ..., J)$ such that*

Kato–Glattfelder(?)

$V = B^* A$

$$(\varphi, V\psi) = \sum_{j=1}^{J} (B_j \varphi, A_j \psi)\,, \quad \varphi, \psi \in X^*\,.$$

Moreover, considered as (unbounded) operators on L^2, the A_j, B_j are closed on some domains satisfying $\mathcal{D}(A_j), \mathcal{D}(B_j) \supseteq H^1(\mathbb{R}^d)$ (which is natural in view of $H^1 \hookrightarrow X^*$.) (Recall Kato smoothing theory in Section 2.)



BS

For instance $V \in \ell^{(d+1)/2} L^{d/2}$ are admissible where

$\frac{d}{2}$, $\frac{d+1}{2}$ $\sim$

$$\|V\|_{\ell^{(d+1)/2} L^{d/2}} = \Big[\sum_s \|V\|_{L^{\frac{d}{2}}(Q_s)}^{\frac{d+1}{2}} \Big]^{\frac{2}{d+1}}$$

and $\{Q_s\}_{s \in \mathbb{Z}^d}$ is a collection of axis-parallel unit cubes such that $\mathbb{R}^d = \bigcup_s Q_s$. The exponent $d/2$ indicates the local integrability which is optimal to realize $-\Delta + V$ self-adjointly, whereas the exponent $(d+1)/2$ indicates its decay. (Note that $2/(d+1) = 1/p_d - 1/p'_d$.)

Theorem 1.3 (Agmon–Hörmander–Kato–Kuroda for admissible V). *Let V be admissible, then*

(1) *$H \equiv -\Delta + V$ defines a self-adjoint operator on $\mathcal{D}(H) = \{u \in H^1 : Hu \in L^2\}$. Moreover, $H \geq -c$ for some $c \in \mathbb{R}$.*

(2) *$\mathcal{E} := \sigma_{pp} \setminus 0$ is discrete in $\mathbb{R} \setminus 0$, i.e., $\mathcal{E} \cap I$ is finite for any compact $I \subseteq \mathbb{R} \setminus 0$. Moreover, each eigenvalue has at most finite multiplicity.*

(3) *The eigenfunctions u of H decay rapidly, i.e., for each $N \geq 0$, one has that $< x >^N u \in H^1(\mathbb{R}^d)$.*

(4) *We have a LAP for H, i.e., for compact $I \subseteq (\mathbb{R} \setminus 0) \setminus \mathcal{E}$, we have*

$$\sup_{\lambda \in I,\, \varepsilon \in (-1,1) \setminus 0} \|(-\Delta + V - (\lambda \pm i\varepsilon))^{-1}\|_{X \to X^*} \lesssim_{V,I} 1\,.$$

In particular, $\sigma(H) \cap I = \sigma_{ac}(H) \cap I$, i.e., the spectrum of H on I is purely ac.

(5) *$\sigma_{sc}(H) = 0$ and $\sigma_{ac}(H) = \sigma_{ac}(-\Delta) = [0, \infty)$.*

(6) *The generalized wave operators $\Omega_\pm(H, H_0) := s - \lim_{t \mp \infty} \mathrm{e}^{itH} \mathrm{e}^{-itH_0} P_{ac}^{(0)}$ and $\Omega_\pm(H_0, H) := s - \lim_{t \mp \infty} \mathrm{e}^{itH_0} \mathrm{e}^{-itH} P_{ac}^{(V)}$ exist and are complete.*

Remark (Embedded eigenvalues). *In principle the theorem does not rule out point spectrum in $[0, \infty)$. However, a deep result by Koch and Tataru [8, Theorem 3] actually*

$$\underline{(H-z)^{-1}} = \underline{\left(\mathbb{1} + R_0(z)V\right)^{-1}} \, \underline{R_0(z)}$$

Lippmann-Schwinger-

4.1

a) $\mathbb{C} \setminus [0,\infty) \to \mathcal{L}(X, X^*)$ ist analytisch

$z \mapsto R_0(z)$

b) $\forall \lambda \in (0,\infty)$, existieren Operatoren $R_0(\lambda + i0)$

$$\| R_0(\lambda + i0) \|_{X \to X^*} \lesssim 1$$

$$(R_0(\lambda+i\epsilon)g, \phi) \xrightarrow{\epsilon \to 0} (R_0(\lambda+i0)g, \phi), \quad g \in X, \ \phi \in \mathcal{S}$$

$$\| \mathbb{1}_{|x|<R} [R_0(\lambda+i\epsilon) - R_0(\lambda+i0)] g \| \xrightarrow{\epsilon \to 0} 0$$

$g \in X$.

c) $(-\Delta - \lambda - i\epsilon) R_0(\lambda+i\epsilon) \underset{X}{g} = g$ im $\mathcal{S}'$-Sinne

$g \in X$.

Pf

a) ✓

c) $\lambda < 0$, oder $\epsilon \neq 0$. ✓

$\lambda > 0$, $\epsilon = 0$ folgt aus b)

b) $f \in \mathcal{S}$ $\quad (-\Delta - z)^{-1} f(x) = T_z * f(x)$

$$|(-\Delta - z)^{-1}(x)| \lesssim |x|^{-(d-2)} \mathbb{1}_{|x|<1} + |x|^{-\frac{d-1}{2}} \mathbb{1}_{|x|>1}$$

glm $\operatorname{Im} z$ $\qquad \widehat{d\sigma}(x)$

$T_{\lambda+i\epsilon} * f(x) \xrightarrow{\epsilon\to 0} T_{\lambda+i0} * f(x)$ (majorisiert)

$$\|R_0(\lambda+i0)f\|_{X^*} \le \lim_{\epsilon\to 0} \|T_{\lambda+i\epsilon} * f\|_{X^*} \overset{4.1}{\le} \|f\|_X$$

$\to$ FAPI per Stetigkeit auf Fkte in X

Prinzip d. glm. Beschr. $\to$ $\overline{\text{I}}$ hochziehen

$$(R_0(\lambda+i\epsilon)g, \phi) \to (R_0(\lambda+i0)g, \phi) \quad g\in X$$

$g\in\mathcal{S}$ mit maj. Kon. ☐

4.2 $V: X^* \to X$ kompakt

Pf $V = V_1 + V_2$

$V_1 \in C_c$? $\hookrightarrow$ Rellich

V_2: $\|V_2\|_{X^*\to X} < \epsilon$

4.3 $g\in X$

$$\operatorname{Im}(g, R_0(\lambda+i0)g) = c_1 \int_{|\xi|=\sqrt{\lambda}} |\hat g(\xi)|^2 \, d\sigma(\xi)$$

$\phi\in C_c^\infty$ (bzw. X ~ray-Trafo) $\hookrightarrow \langle x\rangle^\sigma$

$$\int |(R_0(\lambda+i0)g)(x)|^2 \phi\left(\frac{x}{R}\right)\frac{dx}{R} \xrightarrow{R\to\infty} c_2(\phi)\int_{|\xi|=\sqrt{\lambda}} |\hat g(\xi)|^2 d\sigma$$

$$d\sigma(\xi) = \frac{d\Sigma(\xi)}{2|\xi|} \quad \text{euklid}$$

ϕ $\int_{-\infty}^{\infty} \phi(x+t\omega)\,dt < \infty \quad \forall\, x\in\mathbb{R}^d,\ \omega\in S^{d-1}$

Pf $g \in \mathcal{S}$ ✓ ($\to$ Hörmander I, II)

$g \in X$ $\to$ mit TS oder Agmon-Hörmander

4.4 $$\underline{\underline{(H-z)^{-1}}} = \underline{\underline{(\mathbb{1} + R_0(z)V)}}^{-1} \underline{\underline{R_0(z)}}$$

$$\mathcal{E} = \sigma_{pp}(H) \setminus 0$$

$$\tilde{\mathcal{E}} = \{\lambda \in \mathbb{R} : \exists f \in X^* : (\mathbb{1}_{X^*} + R_0(\lambda + i0)V) f = 0\}$$

$$\Rightarrow \mathcal{F}_\lambda = \{f \in X^*, (\mathbb{1}_{X^*} + R_0(\lambda+i0)V) f = 0\}$$

Ziel $\tilde{\mathcal{E}} = \mathcal{E}$, $\tilde{\mathcal{E}}$ diskret.

4.5 Sei $u \in \mathcal{F}_\lambda \Rightarrow \| \langle x \rangle^N u \|_{X^*} \lesssim \|u\|_{X^*}$.

Pf $$\| \langle x \rangle^N u \|_{X^*} \overset{1.2}{\lesssim} \| \langle x \rangle^N (\Delta + \lambda) \underbrace{u}_{= -R_0(\lambda+i0)\underset{\in X}{Vu}} \|_X$$

$$\overset{4.1}{=} \| \langle x \rangle^N V u \|_X$$

$$\overset{V \text{ zul.}}{<} \epsilon \| \langle x \rangle^N u \|_{X^*} + a_\epsilon \|u\|_{X^*}$$

$$\lim_{R \to \infty} \int_{|x| < R} \underbrace{|u|^2}_{-R_0 Vu} \frac{dx}{R} \overset{4.3}{=} \int_{|\xi| = \sqrt{\lambda}} d\sigma(\xi) \, |\widehat{Vu}(\xi)|^2$$

$$\overset{4.3}{=} \operatorname{Im} (Vu, \underbrace{R_0(\lambda+i0) Vu}_{= -u})$$

$$= 0$$

$\square$

4.5 $\quad \tilde{\Sigma} \subseteq \Sigma, \quad \tilde{\Sigma}$ diskret $\quad (H^1 \hookrightarrow) X^* \subseteq H^1_{loc}$

Pf $\quad (\underline{\underline{1_{X^*}}} + R_0(\lambda \mp i0) \overbrace{V) u}^{X} = 0, \quad u \in \acute{F}_\lambda$

4.1 $\quad (-\Delta - \lambda) + Vu = 0 \; ; \; u \in H^1 \; ?$

$\hookrightarrow$ 4.4. u fällt schnell.

Ang $\exists \infty$ EW λ_n mit u_n EV.

Beob: $\|u_n\|_{H^1} = 1$

Beh $\quad \exists$ konv. Tf. von u_n in H^1

$$u_n = R_0(-1)\left[(\underline{\lambda_n + 1} - V)u_n\right]$$

$$= (\lambda_n + 1) \underbrace{R_0(-1)\langle x\rangle^{-2}}_{\text{kompakt in } H^1} \underbrace{\langle x\rangle^2 u_n}_{\text{zulässig}} - \underbrace{R_0(-1)V u_n}_{\text{komp in } H^1}$$

$$u_n \xrightarrow{H^1} u_\infty \qquad \|u_\infty\|_{H^1} = 1 \quad ↯$$

$$(u_n, u_m) = \delta_{n,m}. \quad \rightarrow u_\infty = 0$$

4.6 $\quad \|(1_{X^*} + \underbrace{R_0(\lambda + i\epsilon)V}_{\text{kompakt}})^{-1}\|_{X^*, X^*} \lesssim 1, \quad \epsilon \in [0,1], \; \lambda \in \mathbb{R} \setminus 0$

$\rightarrow$ analytisch Fredholm bis auf einer diskreten Mg. in $\mathbb{C}$ invertierbar.

$\underline{\underline{\epsilon = 0}} \quad \tilde{\Sigma} \qquad \epsilon > 0$

Pf 1.3. c) $Hu = \lambda u, \quad u \in \mathcal{D}(H)$

$$\|\langle x\rangle^N u\|_{X^*} \overset{1.2}{\lesssim} \|\langle x\rangle^N \underbrace{(\Delta+\lambda)u}_{-Vu}\|_X$$

$$< \epsilon \|\langle x\rangle^N u\|_{X^*} + a_\epsilon \|u\|_{X^*} \checkmark$$

d) $\underbrace{\sigma_{pp} \setminus 0}_{\mathcal{E}}$ diskret

$\tilde{\mathcal{E}} \subseteq \mathcal{E} \checkmark$

$\uparrow$ diskret

$\mathcal{E} \subseteq \tilde{\mathcal{E}}$

$$(-\Delta-\lambda)u + Vu = 0, \quad u \in H^1.$$

$$\underbrace{R_0(\lambda)(-\Delta-\lambda)u}_{\overset{?}{=} u} + R_0(\lambda)Vu = 0$$

$$R_0(\lambda)\underbrace{(-\Delta-\lambda)\overset{H^1}{u}}_{\in H^{-1} \hookrightarrow X} = u' \in X^* \quad / \; -\Delta-\lambda$$

Lin/ $(-\Delta-\lambda)(u-u') = 0$;

$$\|u-u'\|_{X^*} \overset{1.2}{\leq} \|(\Delta+\lambda)(u-u')\|_X = 0$$

$u \checkmark \quad u \in \mathcal{D}(H)$

$$\lim_{R\to\infty} \int_{|x|<R} |u'(x)|^2 \frac{dx}{R} \overset{4.3}{=} \cdot \int_{|\xi|=|\lambda|} |\overbrace{\widehat{(\Delta+\lambda)u}}^{}(\xi)|^2 d\sigma(\xi)$$

$R_0(-\Delta-\lambda)$

$= 0$ pktweise

$\Rightarrow \hat{u} \in C_\infty$ u fällt schnell

$$(H-z)^{-1} = (\mathbb{1} + \underbrace{R_0(z)V}_{\text{kompakt}})^{-1} \underbrace{R_0(z)}_{\text{LAP } \checkmark}$$

$$\to \quad \|(H-z)^{-1}\|_{X,X^*} \lesssim 1, \qquad z = \lambda + i\epsilon$$

$\epsilon > 0$

$\cancel{\lambda \in (\ldots)}$

$$\Rightarrow \quad \sigma_{sc} = \emptyset \qquad \lambda \in (\mathbb{R} \setminus \mathcal{E}) \setminus 0$$

$$\sigma_{ac} = [0,\infty)$$

$(\varphi, dE(\lambda)\varphi)$

Mapt

$$\int \frac{d\mu(\lambda)}{\lambda - z} = F(z)$$

$\hookrightarrow$ Borel-Transf.

$$(\varphi, \operatorname{Im} R(\lambda+i0)\varphi) < \infty \quad \text{Lon Maße}$$

$\hookrightarrow \varphi \in \mathcal{H}_{ac}$

$$\|(T-z_n)^{-1}\| > \frac{1}{\operatorname{dist}(z_n, \sigma(T))}$$

z_n

$$\mathcal{L}(X \to X^*) \qquad -\Delta, \quad -\Delta + V \ldots \quad \text{glm. in } \underline{\operatorname{Im} z}$$

$$\|(T-\lambda+i\epsilon))^{-1}\|_{X\to X^*} \lesssim 1$$

$\ell^{p} L^{d/2}$

says that for $V \in \ell^{(d+1)/2} L^{d/2}$ there are no embedded eigenvalues. *This is sharp in view of the sequence of counterexamples*[3] *of Ionescu–Jerison* [6] *satisfying*

Ionescu-Jerison

$$L^p(\mathbb{R}^d) \ni |V_n(x)| \lesssim \frac{1}{n + |x'|^2 + |x_d|}\,, \quad p > \frac{d+1}{2} \tag{1.1}$$

and $\lim_{n\to\infty} \|V_n\|_{L^p(\mathbb{R}^d)} = 0$[4]. *Cuenin* [1] *recently related that counterexample to Knapp's counterexample in* $\|\hat{f}|_S\|_{L^q} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$ *which gave* $1/q \geq (d+1)/(p'(d-1))$. *Let* $a \in \mathbb{R}^d$, $\delta \ll 1$, *and take* $f(x) = \chi_{\{|x'-a'|<\delta^{-1},\, |x_d-a_d|<\delta^{-2}\}}(x)$. *Then* $\hat{f}$ *is morally supported on the dual rectangle* $R_\delta^* = \{|\xi'| < \delta,\, |\xi_d| < \delta^2\}$[5]. *Now pave* $\mathbb{R}^d \setminus B_0(1)$ *with finitely overlapping rectangles of dimension* $(2^j)^{d-1} \times 4^j$, *let* $\mathbf{1}_{\mathbb{R}^d \setminus B_0(1)}(x) = \sum_{j\geq 1} \chi_j(x)$, *and* $u(x) = \sum_{j\geq 1} 4^{-Nj}\chi_j(x)$ *(being a superposition of Knapp examples). Then* $|u(x)| \sim (|x'|^2 + |x_d|)^{-N}$ *and, since* ΨDO*s do not move the support too much,*

$$T(D)u(x) = \sum_{j\geq 1} 4^{-Nj}(T|_{R_j^*} + O(4^{-j}))\chi_j(x) \sim \lambda u(x) + O((|x'|^2 + |x_d|)^{-N-1})$$

where we expanded $T(\xi) = \lambda + (\partial_d T(0))\xi_d + O(|\xi|^2)$ *around the aspired eigenvalue* $\lambda \in \mathbb{C}$. *Thus,* $V = -(T(D) - \lambda)u/u$ *is smooth and sastisfies pointwise bound in* (1.1).

Lemma 4.1 (Analyticity of free resolvent and existence of boundary values)**.**

(1) *The map*

$$\begin{aligned} \mathbb{C} \setminus [0,\infty) &\to \mathfrak{B}(X, X^*) \\ z &\mapsto R_0(z) \end{aligned}$$

is analytic.

(2) *For any* $\lambda \in (0,\infty)$ *there are operators* $R_0(\lambda \pm i0) \in \mathfrak{B}(X, X^*)$ *such that*

$$\|R_0(\lambda \pm i0)\|_{X\to X^*} \lesssim_\delta 1\,, \quad \lambda \in (\delta, \delta^{-1}),\ \delta \in (0,1)\,.$$

Moreover, for any sequences $(\lambda_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ *and* $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq [0,\infty)$ *with* $\lambda_n \to \lambda$ *and* $\varepsilon_n \to 0$, *we have*

$$\begin{aligned} (R_0(\lambda_n \pm i\varepsilon_n)f, \varphi) &\to (R_0(\lambda \pm i0)f, \varphi)\,, \quad f \in X,\ \varphi \in \mathcal{S} \\ \|\mathbf{1}_{|x|\leq R}[R_0(\lambda_n \pm i\varepsilon_n) - R_0(\lambda \pm i0)]f\| &\to 0\,, \quad f \in X,\ R \geq 1\,. \end{aligned}$$

(3) *For* $\lambda \in \mathbb{R} \setminus 0$ *and* $\varepsilon \geq 0$, *we have*

$$(-\Delta - (\lambda \pm i\varepsilon))R_0(\lambda \pm i\varepsilon)g = g\,, \quad g \in X$$

in distributional sense, i.e., whenever the inner product with $\mathcal{S}$ *functions is taken.*

Lemma 4.2. *If* V *is admissible, then* $V : X^* \to X$ *is compact.*

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[3] Their construction is analogous to that of Wigner and von Neumann [10].

[4] This is interesting in view of the Laptev–Safronov conjecture, cf. Frank–Simon [2].

[5] A Knapp example living at the south pole $\xi_d \sim 0$.

The following establishes the link between Fourier restriction and the existence of boundary values of the free resolvent. See also Hörmander [3, Theorem 14.2.2 and Corollary 14.3.10] and Section 3.

Lemma 4.3. *Let $\Phi \in C_c^\infty(\mathbb{R}^d)$ with $\Phi(0) = 1$ and $\Phi(x) = 0$ whenever $|x| \geq 1$. Then for any $\lambda > 0$ and $g \in X$, we have*

$$\Im(g, R_0(\lambda \pm i0)g) = c_1(\lambda, \pm) \int_{\sqrt{\lambda}\mathbb{S}^{d-1}} |\hat{g}(\xi)|^2 \, d\sigma(\xi)$$

$$\lim_{R\to\infty} \int_{\mathbb{R}^d} |(R_0(\lambda \pm i0)g)(x)|^2 \Phi(\frac{x}{R}) \frac{dx}{R} = c_2(\lambda, \Phi, \pm) \int_{\sqrt{\lambda}\mathbb{S}^{d-1}} |\hat{g}(\xi)|^2 \, d\sigma(\xi)$$

where $d\sigma(\xi)$ denotes the "canonical" surface measure, i.e., $d\sigma(\xi) = d\Sigma(\xi)/(2|\xi|)$ where $d\Sigma(\xi)$ denotes the euclidean (Lebesgue) surface measure.

Recall the Leray measure $P^*\delta_0 = d\Sigma(\xi)/|\nabla P(\xi)|$ whenever $P : \mathbb{R}^d \to \mathbb{R}$ with $\nabla P \neq 0$, $\delta_0 \in \mathcal{E}'(\mathbb{R}^d)$ is the usual d-dimensional Dirac distribution at the origin, and $d\Sigma$ is the euclidean surface measure on $\{\xi \in \mathbb{R}^d : P(\xi) = 0\}$, see also Hörmander [4, Theorem 6.1.5].

Next, the relation between eigenfunctions of H and solutions to $(\mathbf{1}_{X^*} + R_0(\lambda \pm i0)V)f = 0$ (the *"Lippmann–Schwinger equation"*) is discussed. Let

$$\tilde{\mathcal{E}}^\pm = \{\lambda \in \mathbb{R} \setminus 0 : \exists f \in X^* \setminus 0 \text{ s.t. } (\mathbf{1}_{X^*} + R_0(\lambda \pm i0)V)f = 0\} = \tilde{\mathcal{E}}$$
$$\mathcal{F}_\lambda^\pm = \{f \in X^* : (\mathbf{1}_{X^*} + R_0(\lambda \pm i0)V)f = 0\}\,.$$

Our goal is to show $\tilde{\mathcal{E}} = \mathcal{E} := \sigma_{pp}(H) \setminus 0$.

Lemma 4.4 (Rapid decay of solutions to "Lippmann–Schwinger")**.** *Let $\lambda \in \tilde{\mathcal{E}}$ with corresponding $f \in F_\lambda^\pm$. Then for any $N \geq 0$, we have*

$$\| < x >^{2N} f\|_{X^*} \lesssim_{N,V,\lambda} \|f\|_{X^*}\,.$$

Let $\mathfrak{H}_\lambda = \{u \in \mathcal{D}(H) : Hu = \lambda u\}$ denote the vector space of eigenfunctions of H.

Lemma 4.5 (Preliminary relationship between $\mathcal{E}$ and $\tilde{\mathcal{E}}$)**.**

(1) *For any $\lambda \in \mathbb{R} \setminus 0$ we have $\mathcal{F}_\lambda^+ \cup \mathcal{F}_\lambda^- \subseteq \mathfrak{H}_\lambda$ and so $\tilde{\mathcal{E}} \subseteq \mathcal{E}$.*
(2) *$\tilde{\mathcal{E}}$ is discrete in $\mathbb{R} \setminus 0$, i.e., $I \cap \tilde{\mathcal{E}}$ is finite for any compact $I \subseteq \mathbb{R} \setminus 0$.*
(3) *For any $\lambda \in \mathbb{R} \setminus 0$ the vector spaces $\mathcal{F}_\lambda^\pm$ are finite-dimensional.*

Lemma 4.6 (Uniform invertibility of $\mathbf{1}_{X^*} + R_0(\lambda \pm i\varepsilon)V$ away from $\tilde{\mathcal{E}}$)**.** *For any $\lambda \in (\mathbb{R} \setminus 0) \setminus \tilde{\mathcal{E}}$ the operators $\mathbf{1}_{X^*} + R_0(\lambda \pm i\varepsilon)V$ are invertible on X^* with*

$$\sup_{\lambda\in I} \sup_{\varepsilon\in[0,1]} \|(\mathbf{1}_{X^*} + R_0(\lambda \pm i\varepsilon)V)^{-1}\|_{X^*\to X} \lesssim_I 1\,, \quad I \subseteq (\mathbb{R} \setminus 0) \setminus \tilde{\mathcal{E}} \text{ compact}.$$

Lemma 5.1 (Resolvent formula for $(H - z)^{-1}$ away from $\mathbb{R}$)**.** *For any $\lambda \in \mathbb{R}$ and $\varepsilon \in \mathbb{R} \setminus 0$, the operator*

$$\tilde{R}_V(\lambda + i\varepsilon) := (\mathbf{1}_{H^1} + R_0(\lambda \pm i\varepsilon)V)^{-1} R_0(\lambda + i\varepsilon) : L^2 \to \mathcal{D}(H)\,.$$

is well defined and bounded. Moreover, it is a right inverse, i.e.,

$$[H-(\lambda+i\varepsilon)]\tilde{R}_V(\lambda+i\varepsilon)=\mathbf{1}_{L^2}\,.$$

2. Elements of Kato smoothing theory

We recall some classic facts, cf. [9, Section XIII.7].

Definition (*H*-smoothness)**.** *Let A be closed and H self-adjoint. Then A is* H-smooth *if and only if*

$$\sup_{\varepsilon>0,\|\varphi\|=1}\int_{\mathbb{R}}(\|A\underbrace{R(\lambda+i\varepsilon)}_{e^{itH}}\varphi\|^2+\|AR(\lambda-i\varepsilon)\varphi\|^2)\,d\lambda<\infty$$

or

$$\sup_{\mu\notin\mathbb{R},\|\varphi\|=1}\|AR(\mu)\varphi\|^2\cdot|\Im(\mu)|<\infty\,.$$

See also [9, Theorem XIII.25] for further characterizations of H-smoothness. Recall that if

$$\boxed{H=H_0+\sum_{j=1}^{J}B_j^*A_j}$$

and A_j are H_0-smooth and B_j are H-smooth, then the $\Omega_\pm$ exist *and are unitary* [9, Theorem XIII.24]. This assumption on $V=\sum_{j=1}^J B_j^*A_j$ leads to a boring situation as it rules out $\sigma_{pp}(H)$ (if $\sigma(H_0)$ is purely ac, e.g.), so we will relax it now.

Definition (Local H-smoothness)**.** *A is called* H-smooth on a Borel set $I\subseteq\mathbb{R}$ *if AP_I is H-smooth. (Here P_I denotes the PVM associated to H.)*

Theorem XIII.30 (Sufficient criteria for local H-smoothness)**.** *Suppose that either*

(1) $\sup_{\lambda\in I}\sup_{0<|\varepsilon|<1}|\varepsilon|\|AR(\lambda+i\varepsilon)\|^2<\infty$ *or*
(2) $\sup_{\lambda\in I}\sup_{0<|\varepsilon|<1}\|AR(\lambda+i\varepsilon)A^*\|<\infty$

hold. Then A is H-smooth on $\overline{I}$.

$\|AR_V(\lambda+i\epsilon)\|_{L^2,L^2}$

$X^*\to L^2$

Define the *local wave operators*

$\|R_V(\lambda+i\epsilon)f\|_{L^2}^2=(f,R_V(\lambda+i\epsilon)^*R_V(\lambda+i\epsilon)f)$

$$W_\pm=s-\lim_{t\to\mp\infty}\mathrm{e}^{itH}\mathrm{e}^{-itH_0}P_I^{(0)}\,,\qquad\widetilde{W}_\pm=s-\lim_{t\to\mp\infty}\mathrm{e}^{itH_0}\mathrm{e}^{-itH}P_I^{(V)}\,.$$

$=\frac{1}{2i\epsilon}(f,R_V(\lambda-i\epsilon)-R_V(\lambda+i\epsilon)f)$

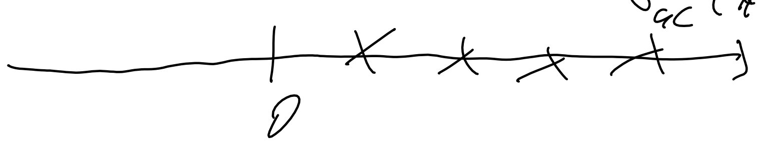
Theorem XIII.31 (Existence and completeness of local wave operators)**.** *If $H=H_0+\sum B_j^*A_j$ and A_j are H_0-smooth and B_j are H-smooth on some open interval $I\subseteq\mathbb{R}$, then the local wave operators $W_\pm$ and $\widetilde{W}_\pm$ exist and satisfy*

$\le\|f\|_X\underbrace{\|R_Vf\|_{X^*}}_{\le\|f\|_X\ \checkmark}$

$$W_\pm^*=\widetilde{W}_\pm\,,\qquad\widetilde{W}_\pm W_\pm=P_I^{(0)}\,,\qquad W_\pm\widetilde{W}_\pm=P_I^{(V)}\,.$$

Corollary (Paving large sets)**.** *Let $S\subseteq\mathbb{R}$ with $S=\bigcup_{\ell\geq1}I_\ell$ where I_ℓ are open bounded intervals and $H=H_0+\sum_j B_j^*A_j$. Suppose A_j are H_0-smooth and B_j are H-smooth on I_ℓ and that $\sigma(H_0)\setminus S$ and $\sigma(H)\setminus S$ have zero Lebesgue measure. Then the generalized wave operators $s-\lim_{t\to\mp\infty}\mathrm{e}^{itH}\mathrm{e}^{-itH_0}P_{ac}^{(0)}$ exist and are complete.*

$\sigma_{ac}(H)=\sigma_{ac}(H_0)$ $\qquad S=[0,\infty)\setminus\mathcal{E}$



3. Distorted Fourier transform

Let

$$H = P_0(D) + V(x, D) \quad \text{in } L^2(\mathbb{R}^d)$$

where P_0 is real and simply characteristic[6] (see Hörmander [3, Definition 14.3.1]), $\sigma_{pp}(P_0) = \{0\}$, and $V(x, D)$ is a symmetric short range perturbation of P_0 in the sense of Hörmander [3, Definition 14.4.1]. Recall the Agmon–Hörmander spaces B and B^* and let

$$\begin{aligned} Z(P_0) &:= \{\lambda \in \mathbb{R} : \ P_0(\xi) = \lambda \text{ and } dP_0(\xi) = 0 \text{ for some } \xi \in \mathbb{R}^d\} \quad \text{and} \\ S_\lambda &:= \{\xi \in \mathbb{R}^d : P_0(\xi) = \lambda\}\,. \end{aligned}$$

Recall

$$\begin{aligned} \int \mathbf{1}_\Omega(\lambda)(dE_\lambda^{(0)} f, f) &= \pm \lim_{\varepsilon \searrow 0} \frac{1}{\pi} \int_{\mathbb{R}} \mathbf{1}_\Omega(\lambda) \Im (R_0(\lambda \pm i\varepsilon) f, f)\, d\lambda \\ &= \int_{\mathbb{R}} d\lambda\, \mathbf{1}_\Omega(\lambda) \int_{S_\lambda} |\hat{f}(\xi)|^2 \, d\sigma_{S_\lambda}(\xi)\,, \quad f \in L^2 \end{aligned}$$

and the resolvent formula $R(\lambda \pm i0) f = R_0(\lambda \pm i0) f_{\lambda \pm i0}$ where $f_z = (1 + V R_0(z))^{-1} f$ is a continuous function of $z \in \mathbb{C}^\pm \setminus (\sigma_{pp}(H) \cup Z(P_0))$ with values in B. Thus, we have

$$\int \mathbf{1}_\Omega(\lambda)(dE_\lambda^{(V)} f, f) = \int_{\mathbb{R}} d\lambda\, \mathbf{1}_\Omega(\lambda) \int_{S_\lambda} |\hat{f}_{\lambda \pm i0}(\xi)|^2 \, d\sigma_{S_\lambda}(\xi)\,, \quad f \in B\,,$$

whenever $\Omega \cap (\sigma_{pp}(H) \cup Z(P_0)) = \emptyset$. This motivates

Definition 1. *If $f \in B$, then the L^2 functions defined by*

$$\begin{aligned} (\mathcal{F}_\pm f)(\xi) &= \mathcal{F}[(1 + V R_0(\lambda \pm i0))^{-1} f](\xi)\,, \quad \xi \in S_\lambda \\ &= \mathcal{F}[(1 - V R(\lambda \pm i0)) f](\xi) \end{aligned} \tag{3.1}$$

almost everywhere in S_λ are called distorted Fourier transforms of *f.*

We recall the following properties of solutions of scattering states. Let $B^*_{P_0} = \{u : P_0^{(\alpha)} u \in B^* \text{ for every } \alpha\}$.

Lemma 2 (Hörmander [3, Lemma 14.6.6])**.** *If $u \in B^*_{P_0}$, $\lambda \notin Z(P_0)$, and $(P_0(D) + V - \lambda) u = 0$, then u is given by the solution of the* Lippmann–Schwinger *equation*

$$u = u_\pm - R_0(\lambda \mp i0) V u \tag{3.2}$$

$$= (1 - R(\lambda \mp i0) V) u_\pm\,, \tag{3.3}$$

where

$$\hat{u}_\pm = v_\pm \delta(P_0 - \lambda) = v_\pm d\sigma_{S_\lambda}(\xi)\,, \quad v_\pm \in L^2(S_\lambda, d\Sigma_{S_\lambda})$$

[6] Examples are hypoelliptic operators (operators whose fundamental solutions E have sing supp$(E) = \{0\}$) or operators of real principal type (operators whose principal symbol P_m is real and $P_m'(\xi) \neq 0$ for $\xi \neq 0$

and

$$\int_{S_\lambda} (|v_+|^2 - |v_-|^2)\, d\sigma_{S_\lambda}(\xi) = 0 \tag{3.4}$$

where $d\sigma_{S_\lambda}(\xi) = |\nabla P_0(\xi)|^{-1} d\Sigma_{S_\lambda}(\xi)$ *and* $d\Sigma_{S_\lambda}(\xi)$ *is the euclidean surface measure on* S_λ*. Moreover, if* $\lambda \notin (Z(P_0) \cup \sigma_{pp}(P_0 + V))$*, then*

$$(\mathcal{F}_+ f, \hat{u}_+) = (\mathcal{F}_-, \hat{u}_-) = (f, u)\,, \quad \text{if } f \in B\,. \tag{3.5}$$

Let us also recall

Theorem 3 (Hörmander [3, Lemma 14.6.4 and Theorem 14.6.5])**.** $\mathcal{F}_\pm : E^c L^2(\mathbb{R}^d) \to \widehat{L^2(\mathbb{R}^d)}$ *is an isometric operator, which vanishes on* $E^{pp} L^2(\mathbb{R}^d)$*, with*

$$\|E^c f\|_2^2 = \int_{\mathbb{R}^d} |\mathcal{F}_\pm f(\xi)|^2\, d\xi\,.$$

Moreover, the intertwining property

$$\mathcal{F}_\pm \mathrm{e}^{itH} = \mathrm{e}^{itP_0(\xi)} \mathcal{F}_\pm$$

holds for all $t \in \mathbb{R}$*. In particular, the restriction of* H *to* $E^c L^2$ *is absolutely continuous (since* P_0 *has purely absolutely continuous spectrum).*

Moreover, $\mathcal{F}_\pm : E^c L^2(\mathbb{R}^d) \to \widehat{L^2(\mathbb{R}^d)}$ *is actually unitary, i.e., the restriction of* H *to* $E^c L^2$ *is unitarily equivalent to* P_0*, i.e.,* $\sigma_c(H) = \sigma_{ac}(H) = \sigma(P_0)$*. In particular, for* $f \in E^c(L^2(\mathbb{R}^d))$*, we have*

$$(\mathcal{F}_\pm H f)(\xi) = P_0(\xi)(\mathcal{F}_\pm f)(\xi)\,, \quad \text{i.e.,} \quad (Hf)(x) = (\mathcal{F}_\pm^* P_0(\cdot) \mathcal{F}_\pm f)(x)\,.$$

In particular, it follows that

$$\mathcal{F}_\pm^* \mathcal{F}_\pm = E^c \quad \text{and} \quad \mathcal{F}_\pm \mathcal{F}_\pm^* = \mathbf{1}_{\widehat{L^2}}\,.$$

The distorted Fourier transform (3.1) can be conveniently represented using the solutions $\varphi_{\xi(\lambda)}(x)$ (for $\xi(\lambda) \in S_\lambda$) of the Lippmann–Schwinger equation (3.2). In fact, we have (see also Ikebe [5] and Yafaev [11, Sections 6.6-6.8])

$$(\mathcal{F}_\pm f)(\xi) = \langle \varphi_\xi, f \rangle\,, \quad \xi \in \bigcup_{\lambda \in \sigma_{ac}(H)} S_\lambda \tag{3.6}$$

$$(\mathcal{F}_\pm^* g)(x) = \int_{\mathbb{R}^d} \varphi_\xi(x) g(\xi)\, d\xi = \int_{\sigma_{ac}(H)} d\lambda \int_{S_\lambda} d\sigma_{S_\lambda}(\xi)\; \varphi_\xi(x) g(\xi)\,. \tag{3.7}$$

Moreover, we have the following expansion theorem (see also Ikebe [5, Theorem 5])

$$f = \sum_{\lambda \in \sigma_{pp}(H)} |\psi_\lambda\rangle\langle\psi_\lambda, f\rangle + \int_{\mathbb{R}^d} |\varphi_\xi\rangle\langle\varphi_\xi, f\rangle\, d\xi \tag{3.8}$$

where $\{\psi_\lambda\}_{\lambda \in \sigma_{pp}(H)}$ denote the L^2-normalized eigenfunctions of H, i.e., $H\psi_\lambda = \lambda\psi_\lambda$. Moreover,

$$Hf = \sum_{\lambda \in \sigma_{pp}(H)} \lambda |\psi_\lambda\rangle\langle\psi_\lambda, f\rangle + \int_{\mathbb{R}^d} P_0(\xi) |\varphi_\xi\rangle\langle\varphi_\xi, f\rangle\, d\xi\,. \tag{3.9}$$

The above results motivate in particular the following definition of the *distorted Fourier restriction and extension operators*

$$(F_{S_\lambda} f)(\xi) = \langle \varphi_\xi, f \rangle = (\mathcal{F}_\pm f)(\xi)\,, \quad \xi \in S_\lambda \tag{3.10}$$

$$(F^*_{S_\lambda} g)(x) = \int_{S_\lambda} d\sigma_{S_\lambda}(\xi)\ \varphi_{\xi(\lambda)}(x) g(\xi) \tag{3.11}$$

which are defined with respect to the canonical measure $d\sigma_{S_\lambda}$. In particular, we have for any $\Lambda \subseteq \sigma_{ac}(H)$,

$$E_H(\Lambda) = \int_{P_0^{-1}(\Lambda)} |\varphi_\xi\rangle\langle\varphi_\xi|\, d\xi = \int_\Lambda d\lambda \int_{S_\lambda} d\sigma_{S_\lambda}(\xi)\ |\varphi_{\xi(\lambda)}\rangle\langle\varphi_{\xi(\lambda)}| = \int_\Lambda d\lambda\ F^*_{S_\lambda} F_{S_\lambda}$$

in a suitable weak sense and in particular, for $\lambda \in \sigma_{ac}(H)$,

$$\frac{dE_H(\lambda)}{d\lambda} = \int_{S_\lambda} d\sigma_{S_\lambda}(\xi)\ |\varphi_{\xi(\lambda)}\rangle\langle\varphi_{\xi(\lambda)}| = F^*_{S_\lambda} F_{S_\lambda}\,.$$

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