Some more sufficient details necessary to prove Thm 3.2.9 ①

Thm 3.2.9 Let μ be a complex or finite measure on $\mathbb{R}$ with Borel transform $F_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{t-z}$. Then the limit $F_\mu(x) := \lim_{\epsilon \to 0} F_\mu(x+i\epsilon)$ exists and is finite for Lebesgue a.e. $x \in \mathbb{R}$

~~Lemma~~ 1 Poisson transforms of measures

Notation $P_\mu(z) = \operatorname{Im} F_\mu(z)$ and $I(x,r) = (x-r, x+r)$ (an interval)

Lemma 1 Let μ be a positive measure. Then for all $x \in \mathbb{R}$ and $y > 0$,

$$\frac{1}{y} P_\mu(x+iy) = \int_0^{1/y^2} \mu\left(I\left(x, \sqrt{u^{-1}-y^2}\right)\right) du$$

Pf RHS equals

$$\int_0^{1/y^2} du \int_{\mathbb{R}} d\mu(t)\, \mathbb{1}_{I(x,\sqrt{u^{-1}-y^2})}(t) = \int_{\mathbb{R}} d\mu(t) \int_0^{1/y^2} du\, \mathbb{1}_{I(x,\sqrt{u^{-1}-y^2})}(t) \quad (*)$$

Since $|x-t| < \sqrt{u^{-1}-y^2} \iff 0 \le u < ((x-t)^2+y^2)^{-1}$, we have

$$\mathbb{1}_{I(x,\sqrt{u^{-1}-y^2})}(t) = \mathbb{1}_{[0,((x-t)^2+y^2)^{-1})}(u)$$

and so $(*) = \int ((x-t)^2+y^2)^{-1} d\mu(t) = \frac{1}{y} P_\mu(x+iy)$ ◻

Lemma 2 Let ν be a complex and μ be a positive measure. Then for all $x \in \mathbb{R}$ and $y > 0$, we have

$$\frac{|P_\nu(x+iy)|}{P_\mu(x+iy)} \le M_{\nu,\mu}(x) := \sup_{r>0} \frac{|\nu|(I(x,r))}{\mu(I(x,r))}, \quad x \in \operatorname{supp}\mu$$

(maximal fct)

Pf Since $|P_\nu| \le P_{|\nu|}$, we may assume wlog that $\nu \ge 0$. Moreover wlog $x \in \operatorname{supp}\mu$, otherwise $M_{\nu,\mu}(x) = \infty$ and there is nothing to prove.

Since

$$\underbrace{\int_0^{1/y^2} \nu\left(I(x,\sqrt{u^{-1}-y^2})\right) du}_{\overset{\text{Lem 1}}{=} \frac{1}{y} P_\nu(x+iy)} = \int_0^{1/y^2} \underbrace{\frac{\nu\left(I(x,\sqrt{u^{-1}-y^2})\right)}{\mu\left(I(x,\sqrt{u^{-1}-y^2})\right)}}_{\le M_{\nu,\mu}(x)} \mu\left(I(x,\sqrt{u^{-1}-y^2})\right) du$$

$$\overset{\text{Lem 1}}{\le} M_{\nu,\mu}(x) \cdot \frac{1}{y} P_\mu(x+iy)$$

◻

<u>Lem 3</u> Let μ be a positive measure. Then

$$G(x) = \int \frac{d\mu(t)}{(x-t)^2} = \infty \quad \text{for } \cancel{\text{a.e. } x \in \mathbb{R}} \quad \mu\text{-a.e. } x$$

Compare with p. 24, especially Thm 3.2.8!

<u>Pf</u> HW (we will not need it in the subsequent analysis anyway)

<u>Lem 4</u> Let μ be a positive measure and $f \in C_c(\mathbb{R})$. Then

$$\lim_{\epsilon \to 0} \frac{P_{f\mu}(x+i\epsilon)}{P_\mu(x+i\epsilon)} = f(x) \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}.$$

<u>Remark</u> In fact, the limit holds for all x for which the assertion in Lemma 3 holds. If, e.g., $\mu =$ Lebesgue, then the limit holds for all $x \in \mathbb{R}$.

<u>Pf</u> Note $(*) = \left| \frac{P_{f\mu}(x+i\epsilon)}{P_\mu(x+i\epsilon)} - f(x) \right| \leq \frac{P_{|f-f(x)|\mu}(x+i\epsilon)}{P_\mu(x+i\epsilon)}$. Now fix $\tilde{\epsilon} > 0$ and let $\delta > 0$ be such that $|x-t| < \delta \Rightarrow |f(x) - f(t)| < \tilde{\epsilon}$. Let $M := \sup|f(t)|$ and $C = 2M \int_{|x-t|>\delta} \frac{d\mu(t)}{(x-t)^2}$

$$\Rightarrow P_{|f-f(x)|\mu}(x+i\epsilon) \leq \tilde{\epsilon} P_\mu(x+i\epsilon) + C\epsilon$$

$$\Rightarrow (*) \leq \tilde{\epsilon} + \frac{C\epsilon}{P_\mu(x+i\epsilon)}$$

Now let x be such that $G(x) = \int \frac{d\mu(t)}{(t-x)^2} = \infty$.

Then by monotone convergence, $\lim_{\epsilon \to 0} \frac{\epsilon}{P_\mu(x+i\epsilon)} = \left(\int \frac{d\mu}{(x-t)^2} \right)^{-1} = 0$

and so for all $\tilde{\epsilon} > 0$, $\overline{\lim_{\epsilon \to 0}} \left| \frac{P_{f\mu}(x+i\epsilon)}{P_\mu(x+i\epsilon)} - f(x) \right| \leq \tilde{\epsilon}$ ∎ (upon letting $\tilde{\epsilon} \to 0$)

Thm 5 Let ν be a complex and μ be a pos. measure. Then
$= f\mu + \nu_s$

(1) $$\lim_{\epsilon\to 0} \frac{P_\nu(x+i\epsilon)}{P_\mu(x+i\epsilon)} = f(x) \quad \text{for } \mu\text{-a.e. } x$$

In particular, $\mu \perp \nu \Leftrightarrow \lim_{\epsilon\to 0} \frac{P_\nu(x+i\epsilon)}{P_\mu(x+i\epsilon} = 0$ for μ-a.e. x

(2) If $\nu \geq 0$, then $$\lim_{\epsilon\to 0} \frac{P_\nu(x+i\epsilon)}{P_\mu(x+i\epsilon)} = \infty \quad \text{for } \nu_s\text{-a.e. } x$$

Pf (1) Step 1: Assume $\nu \ll \mu$, i.e., $\nu = f\mu$. Let $g_n \in C_c$ s.t. $\int |f-g_n| d\mu < \frac{1}{n}$ and set $h_n := f - g_n$. Then

$$\left| \frac{P_{f\mu}(x+i\epsilon)}{P_\mu(x+i\epsilon)} - f(x) \right| \leq \underbrace{\frac{P_{|h_n|\mu}(x+i\epsilon)}{P_\mu(x+iy)}} + \underbrace{\left| \frac{P_{g_n\mu}(x+i\epsilon)}{P_\mu(x+i\epsilon)} - g_n(x) \right|}_{\to 0} + |g_n(x) - f(x)|$$

By Lemma 2 and 4 $\quad \underset{\epsilon\to 0}{\leq} M_{|h_n|\mu,\mu}(x)$

$$\overline{\lim_{\epsilon\to 0}} \left| \frac{P_{f\mu}(x+i\epsilon)}{P_\mu(x+i\epsilon)} - f(x) \right| \leq \underbrace{M_{|h_n|\mu,\mu}(x)}_{\xrightarrow{n\to\infty} 0} + \underbrace{|g_n(x) - f(x)|}_{\xrightarrow{n\to\infty} 0 \ \mu\text{-a.e. } x}$$

(by weak-type inequality for maximal fct.)
(cf. Thm 1.8 and Prop. 1.10 in Jaksic)

(up to a subsequence)

Step 2 We are left to show $\mu \perp \nu \Rightarrow \lim_{\epsilon\to 0} \frac{P_\nu(x+i\epsilon)}{P_\mu(x+i\epsilon)} = 0$ for μ-a.e. x
$\nu \geq 0$
$\int \nu < \infty$

To this end let S be a Borel set with $\mu(S) = 0$ and $\nu(\mathbb{R}\setminus S) = 0$.

$$\Rightarrow \frac{P_\nu(x+i\epsilon)}{P_\nu(x+i\epsilon) + P_\mu(x+i\epsilon)} = \frac{P_{1_S(\nu+\mu)}(x+i\epsilon)}{P_{\nu+\mu}(x+i\epsilon)} \xrightarrow[\text{step 1}]{\epsilon\to 0} 1_S(x) \quad \text{for } \nu+\mu\text{-a.e. } x$$

~~By step 1, $\lim_{\epsilon\to 0}$~~ But since $1_S(x) = 0$ for μ-a.e. x, we have

$$\lim_{\epsilon\to 0} \frac{P_\nu(x+i\epsilon)}{P_\nu(x+i\epsilon) + P_\mu(x+i\epsilon)} = 0 \quad \text{for } \mu\text{-a.e. } x \text{ which implies}$$

$$\lim_{\epsilon\to 0} \frac{P_\nu(x+i\epsilon)}{P_\mu(x+i\epsilon)} = 0 \quad \text{for } \mu\text{-a.e. } x$$

(2) Since $\nu \geq 0$, we have $\nu(I(x,r)) \geq \nu_s(I(x,r))$. ~~For $\nu \perp \mu$~~

Wlog $\nu \perp \mu$ since we only assert something for ν_s - a.e. x.

By $\lim_{\epsilon \to 0} \frac{P_\nu(x+i\epsilon)}{P_\nu(x+i\epsilon)+P_\mu(x+i\epsilon)} = \lim_{\epsilon \to 0} \frac{P_{\mathbb{1}_S(\nu+\mu)}(x+i\epsilon)}{P_{\nu+\mu}(x+i\epsilon)} = \mathbb{1}_S(x)$ for $\mu+\nu$-a.e.

we obtain analogously

$$\lim_{\epsilon \to 0} \frac{P_\nu(x+i\epsilon)}{P_\mu(x+i\epsilon)+P_\nu(x+i\epsilon)} = 1 \text{ for } \nu_s = \nu_\phi \text{ - a.e. } x.$$

$$\Rightarrow \lim_{\epsilon \to 0} \frac{P_\mu(x+i\epsilon)}{P_\nu(x+i\epsilon)} = 0 \text{ for } \nu\text{-a.e. } x$$

2 Poisson representation of harmonic functions

Thm 6 Let $V(z)$ be a positive harmonic fct in $\mathbb{C}_+ \Rightarrow \exists\ c \geq 0$ and a positive measure μ on $\mathbb{R}$ s.t. $V(x+iy) = cy + P_\mu(x+i\epsilon)$ and c and μ are uniquely determined by V. Especially $c = \lim_{y \to \infty} \frac{V(iy)}{y}$

Remark By Thm 5, for $d\mu = f(x)dx + d\mu_s$, we have $f(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} V(x+i\epsilon)$ for Lebesgue a.e. x

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\Gamma = \partial\mathbb{D} = \{z : |z| = 1\}$, $\quad p_z(w) := \operatorname{Re} \frac{w+z}{w-z} = \frac{1-|z|^2}{|w-z|^2}$, $z \in \mathbb{D}$, $w \in \partial\mathbb{D}$

Poisson kernel on $\partial\mathbb{D}$

To prove Thm 6, we will use

Thm 7 Let $U \geq 0$ be harmonic on $\mathbb{D} \Rightarrow \exists$ finite positive Borel measure ν on $\partial\mathbb{D}$ s.t. $\forall z \in \mathbb{D}$, we have Herglotz representation

$$U(z) = \int_{\partial\mathbb{D}} p_z(w)\, d\nu(w).$$

Pf of Thm 6 Let $S : \mathbb{C}_+ \to \mathbb{D}$, $z \mapsto \frac{i-z}{i+z}$ be the usual Möbius transformation between $\mathbb{C}_+$ and $\mathbb{D}$. Note that $S(\mathbb{R}) = \partial\mathbb{D} \setminus \{-1\}$ and $S(\infty) = -1$.

Let $K_z(t) = \frac{y}{(x-t)^2 + y^2}$, $z = x + iy \in \mathbb{C}_+$ be the Poisson kernel on $\mathbb{C}_+$

then $(1+t^2) K_z(t) = p_{S(z)}(S(t))$

Let $T : \mathbb{D} \to \mathbb{C}_+$, $\xi \mapsto \frac{\xi - 1}{i\xi + i} = S^{-1}(\xi)$ be the inverse Möbius transf.

and $U(\xi) = V(T(\xi))$ which is positive and harmonic on $\mathbb{D}$, i.e., by Herglotz representation, there is a finite, positive Borel measure on $\partial\mathbb{D}$ s.t. $U(\xi) = \int_{\partial\mathbb{D}} p_\xi(w)\, d\nu(w)$.

Let ν_T the Borel measure on $\mathbb{R}$ that is induced by ν via the Möbius transf. $T : \mathbb{D} \to \mathbb{C}_+$ which takes $T : \partial\mathbb{D} \setminus \{-1\} \to \mathbb{R}$. Then by change of variables,

$$\int_{\partial\mathbb{D}\setminus\{-1\}} p_\xi(\xi)\, d\nu(w) = \int_{\mathbb{R}} (1+t^2) K_{T(\xi)}(t)\, d\nu_T(t)$$

$\Rightarrow$ For $z \in \mathbb{C}_+$, we have $V(z) = \underbrace{p_{S(z)}(-1)}_{=y} \nu(\{-1\}) + \int_{\mathbb{R}} (1+t^2) K_z(t)\, d\nu_T(t)$

Setting $c = \nu(\{-1\})$ and $d\mu(t) = (1+t^2)\, d\nu_T(t)$, the assertion follows.

3 The Hardy class $H^\infty(\mathbb{C}_+)$

$V \in H^\infty(\mathbb{C}_+) \Leftrightarrow V$ holomorphic in $\mathbb{C}_+$ with $\|V\|_\infty = \sup_{y\in(0,\infty)} \sup_{x\in\mathbb{R}} |V(x+iy)|$

$$= \sup_{z\in\mathbb{C}_+} |V(z)| < \infty.$$

$H^\infty(\mathbb{C}_+)$ is a Banach space with the following two main properties that non-tangential (radial, especially) limits exist and are finite.

Thm 8 Let $V \in H^\infty(\mathbb{C}_+)$, then $V(x) := \lim_{\epsilon\to 0} V(x+i\epsilon)$ exists for Lebesgue a.e. x and $\|V\|_{L^\infty(\mathbb{R},dx)} < \infty$.

Thm 9 Let $0 \not\equiv V \in H^\infty(\mathbb{C}_+)$ and $V(x) = \lim_{\epsilon\to 0} V(x+i\epsilon)$. Then

$$\int \frac{|\log|V(x)||}{1+x^2}\,dx < \infty, \quad \text{i.e., in particular, } V(x) \neq 0 \text{ almost everywhere.}$$

Moreover, for any $\alpha \in \mathbb{C}$, either $V(z) \equiv \alpha$ or $|\{x\in\mathbb{R}: V(x) = \alpha\}| = 0$.

There two properties give the following

Thm 10 Let F be holomorphic in $\mathbb{C}_+$ with $\operatorname{Im} F > 0$. The

(1) ~~h~~ $F(x) := \lim_{\epsilon\to 0} F(x+i\epsilon)$ exists and is finite a.e

(2) If $\alpha\in\mathbb{C}$, then either $F(z) \equiv \alpha$ or $|\{x\in\mathbb{R}: F(x) = \alpha\}| = 0$.

Pf (1) Apply Thm 8 to $(F(z)+i)^{-1}$ ▱

(2) Apply Thm 9 to $(F(z)+i)^{-1} - (\alpha+i)^{-1}$.

Pf of Thm 8 Let $d\nu_y(t) := V(t+iy)\,dt$, then for $f \in L^1(\mathbb{R}, dt)$,

$$\left|\int_{\mathbb{R}} f(t)\,d\nu_y(t)\right| \le \|V\|_\infty \|f\|_{L^1(\mathbb{R})}$$

$\Rightarrow$ The map $\Phi_y(f) := \int f\,d\nu_y$ is a linear functional on $L^1(\mathbb{R},dt)$ with $\|\Phi_y\| \le \|V\|_\infty$ $\forall y>0$. Thus, by Banach-Alaoglu, there is a bounded linear functional Φ on $L^1(\mathbb{R},dt)$ ~~with~~ and a sequence $y_n \searrow$ s.t. for all $f \in L^1(\mathbb{R},dt)$, $\Phi(f) = \lim_{n\to\infty} \Phi_{y_n}(f)$

Now let $V \in L^\infty(\mathbb{R},dt)$ be such that $\Phi(f) = \int_{\mathbb{R}} V(t)f(t)\,dt$ and let $f(t) = \frac{1}{\pi}\cdot\frac{y}{(x-t)^2+y^2}$ then $\Phi_{y_n}(f) = \ldots = V(x+i(y+y_n))$ and so $V(x+iy) = \frac{1}{\pi}\,y\int \frac{V(t)}{(x-t)^2+y^2}\,dt$ upon $n\to\infty$.

The statement follows from Thm 5 ▱

Prop 11 (Jensen's formula)
Let $U(z)$ be holomorphic in $\mathbb{D}$ and $U(0)\neq 0$. Let $r\in(0,1)$ and assume that U has no zeros on the circle $|z|=r$. Let $\{\alpha_j\}_{j=1}^n$ be the zeros of U in the region $|z|<r$, counting multiplicities

$$\Rightarrow \quad |U(0)| \prod_{j=1}^n \frac{r}{|\alpha_j|} = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|U(re^{it})|\,dt\right)$$

Remark Actually, the formula holds even if U does have zeros on $|z|=r$, but we will only need the above elementary version.

Pf Set $V(z)=U(z)\prod_{j=1}^n \frac{r^2-\overline{\alpha_j}z}{r(\alpha_j-z)}$. Then for some $\epsilon>0$, $V(z)$ has no zeros inside the disk $|z|<r+\epsilon$ and the function $\log|V(z)|$ is harmonic in that disk ($\sim$ Thm 13.12 in Rudin; it's just the 2d Green's fct.) Thus, by the mean value property of harmonic fcts., $\log|V(0)|=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|V(re^{i\theta})|\,d\theta$ and substitution of $V(z)=U(z)\prod\frac{r^2-\overline{\alpha_j}z}{r(\alpha_j-z)}$ yields the assertion. ∎

Pf of Thm 9 Setting $U(e^{it})=V(\tan(t/2))$ means

$$\int_{\mathbb{R}} \frac{|\log|V(x)||}{1+x^2}\,dx = \frac{1}{2}\int_{-\pi}^{\pi} |\log|U(e^{it})||\,dt,$$ so we just show finiteness of RHS. Recall the Möbius transformations $S:\mathbb{C}_+\to\mathbb{D}$, $\underset{= S^{-1}}{T}:\mathbb{D}\to\mathbb{C}_+$

and let $\underset{\text{on disk}}{U(z)} = \underset{\text{on } \mathbb{C}_+}{V(T(z))} \Rightarrow U$ holomorphic in $\mathbb{D}$ and $\sup_{z\in\mathbb{D}}|U(z)|<\infty$.

Moreover, by change of variables, $U(re^{i\theta})=\int_{-\pi}^{\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)}U(e^{i\varphi})\,d\varphi$

Then the analog of Thm 5 for the circle asserts $\lim_{r\nearrow 1}U(re^{i\theta})=U(e^{i\theta})$ exists Lebesgue-a.e.

Now, we apply Jensen: if $U(0)=0$, let m be such that $U_m(z)=z^{-m}U(z)$ satisfies $U_m(0)\neq 0$ (if $U(0)\neq 0\to m=0$). Let $r_j\nearrow 1$ be a sequence such that U_m has no zeros on $|z|=r_j$. Set

$$J_{r_j} := \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|U_m(re^{i\theta})|\,d\theta = -m\log r_j + \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|U(re^{i\theta})|\,d\theta.$$

Then, Jensen applied to U_m shows that $\mathcal{J}_{r_i} \le \mathcal{J}_{r_j}$ if $r_i \le r_j$.

Note that $\sup_t \log_+ |U(e^{it})| \le \sup_t |U(e^{it})|$ $\quad$ ($\log_+ x = \max(0, \log x)$
$\log_- x = -\min(0, \log x)$))

Thus, by Fatou's lemma, dominated convergence and $U(re^{i\theta}) \xrightarrow{r \nearrow 1} U(e^{i\theta})$, we obtain $\mathcal{J}_{r_1} + \frac{1}{2\pi}\int_{-\pi}^{\pi} \log_- |U(e^{it})| dt \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |U(e^{it})| < \infty$

$\Rightarrow \int_{-\pi}^{\pi} |\log |U(e^{it})|| dt < \infty$ and the "change of variables" $U(e^{it}) = V(\tan(t/2))$ and the ensuing identity yield the assertion $\boxed{}$

Thm 3.2.9 from notes / Thm 12

Let μ be a complex or finite ~~Borel~~ measure, then $F_\mu(x) := \lim_{\epsilon \to 0} F_\mu(x + i\epsilon)$ exists and is finite for Lebesgue-almost every $x \in \mathbb{R}$

Remark It is possible that $\mu \ne 0$ but $F_\mu \equiv 0$; example $d\mu = \frac{dx}{(x-i)(x-2i)}$

Pf We first show $F = \frac{R(z)}{G(z)}$ where $R, G \in H^\infty(\mathbb{C}_+)$ and G has no zeros in $\mathbb{C}_+$.
If $\mu \ge 0$, set $G(z) = \frac{1}{i + F_\mu(z)}$ $\rightarrow$ G holomorphic in $\mathbb{C}_+$, $|G(z)| \le 1$ (since $\operatorname{Im} F_\mu > 0$)

and $F_\mu(z) = \frac{1 - iG(z)}{G(z)}$

If μ is complex, then polarize, i.e., decompose $\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4)$ where the $\mu_j \ge 0$ and decompose analogously $F_\mu = F_{\mu_1} - F_{\mu_2} + i(F_{\mu_3} - F_{\mu_4})$.
By Thms 8 + 9, we know that the limits $R(x) = \lim R(x + i\epsilon)$ and ~~$G(x) = G$~~
$G(x) = \lim G(x + i\epsilon)$

exist and are finite and are non-zero almost everywhere.
$\rightarrow F_\mu(x) = \lim F_\mu(x + i\epsilon) = \lim \frac{R(x+i\epsilon)}{G(x+i\epsilon)}$ satisfies claimed properties $\boxed{}$