Results on H^p and boundary values of analytic fcts on $\mathbb{D}$

(Simon – Harmonic Analysis – Ch. 5)

Q₁ Suppose f analytic on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. What can be said about its boundary values on $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$?

1 Defs

→ Hardy Spaces H^p, $0 < p \leq \infty$ consisting of analytic fcts on $\mathbb{D}$ s.t.

$$\|f\|_p \equiv \sup_{0<r<1} \underbrace{\left(\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p}}_{M_r^{(p)}(f)} < \infty, \quad 0 < p < \infty$$

$d\theta = d\theta / 2\pi$

$$\|f\|_\infty \equiv \sup_{0<r<1} \underbrace{\sup_{\theta \in [0,2\pi)} |f(re^{i\theta})|}_{M_r^{(\infty)}(f)} < \infty \quad p = \infty$$

→ Nevanlinna class N $\quad \sup_{0<r<1} \int_0^{2\pi} \log_+ |f(re^{i\theta})| \, d\theta \equiv \|f\|_N < \infty$

$(\log_+(y) = \max(0, \log y))$

Since for $y > 0$, $\max(1, y) \leq e^y \Rightarrow \log_+ |y| < |y| \Rightarrow \log_+ |y| \leq \frac{|y|^p}{p}$

$\Rightarrow$ for $0 < p < q \leq \infty$ we have $H^q \subseteq H^p \subseteq N$

$\|f\|_p \leq \|f\|_q$ and $\|f\|_N \leq p^{-1} \|f\|_p$

Since real and imaginary part of holomorphic fcts are harmonic, consider also real-valued fcts $u : \mathbb{D} \to \mathbb{R}$. If $\|u\|_p < \infty \Rightarrow u \in h^p$ ($1 \leq p \leq \infty$ here)

If $u \in h^p$, then there is a unique fct $v : \mathbb{D} \to \mathbb{R}$, called the harmonic conjugate, ~~s.t.~~ with $v(0) = 0$ s.t. $u + iv$ is holomorphic on $\mathbb{D}$.

Interesting facts:

- $\|f\|_p = \sup_{0<r<1} (\ldots)^{1/p} = \lim_{r \to 1} (\ldots)^{1/p}$ and $\sup_{0<r<1} \int \log_+ |f(re^{i\theta})| \, d\theta = \lim_{r \to 1} (\ldots)$
- $f \in N \Leftrightarrow f = g/h$ for $g, h \in H^\infty$ with $h \neq 0$ on $\mathbb{D}$
- $\exists f^* \in L^p(\partial\mathbb{D}, d\theta)$ s.t. $\|\underbrace{f_r}_{= f(re^{i\theta})} - f^*\|_{L^p(\partial\mathbb{D}, d\theta)} \xrightarrow{r \to 1} 0$, i.e., $f_r(e^{i\theta}) \to f^*(e^{i\theta})$ for a.e. $\theta \in [0, 2\pi]$
- $f \in H^p \Leftrightarrow f = B g$ and $g \in H^p$ and $\|f\|_p = \|g\|_p$ (non-vanishing on $\mathbb{D}$; B: Blaschke product)
- $1 \leq p \leq \infty \Rightarrow H^p$ and h^p with $\|\cdot\|_p$ are Banach

Lemma 1 Let $f \in N$ with $f \not\equiv 0$ and suppose $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ exists in $\mathbb{C}$ for a.e. θ. Then $\int \log_- |f^*(e^{i\theta})|\, d\theta < \infty$ and in particular, $f^*(e^{i\theta}) \neq 0$ for a.e. θ

Thm 2 (Fatou) Let $f \in N$, $f \not\equiv 0$. Then there is f^* on $\partial\mathbb{D}$ s.t. for a.e. $\theta \in [0, 2\pi)$

a) $\lim_{r\to 1} f(re^{i\theta}) = f^*(e^{i\theta})$

b) $f^*(e^{i\theta}) \neq 0$

Corollary 3 Let $f, g \in N$. If $f^*(e^{i\theta}) = g^*(e^{i\theta})$ for a.e. θ, then $f = g$.

Thm 4 Let f be holomorphic on $\mathbb{D}$ with $f(z) = \sum_{n\geq 0} a_n z^n$. Then

a) $f \in H^2 \Leftrightarrow \sum_{n\geq 0} |a_n|^2 < \infty$ and in this case $\|f\|_2^2 = \sum_{n\geq 0} |a_n|^2$

b) Let $f \in H^2$ and define $f^* \in L^2(\partial\mathbb{D}, d\theta)$ by $f^*(e^{i\theta}) = \sum_{n\geq 0} a_n e^{in\theta}$ as a series in $L^2(\partial\mathbb{D})$. (I.e. a Fourier series)

$\Rightarrow \|f_r - f^*\|_2 \xrightarrow{r\to 1} 0$ where $f_r(e^{i\theta}) = f(re^{i\theta})$

c) Recall Poisson kernel $P_r(\theta) = \operatorname{Re} \frac{1+re^{i\theta}}{1-re^{i\theta}} = \frac{1-r^2}{1+r^2-2r\cos\theta}$.

If f_r and f^* are as in b), then $f_r = P_r * f^*$, i.e.,

$$f_r(e^{i\theta}) = f(re^{i\theta}) = \int_0^{2\pi} \frac{d\varphi}{2\pi} P_r(\theta-\varphi) f^*(e^{i\varphi})$$

and $$f(z) = i \operatorname{Im} f(0) + \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} \operatorname{Re} f^*(e^{i\varphi})$$

d) Define radial maximal fct of f by $(M_{rad} f)(e^{i\theta}) = \sup_{0<r<1} |f(re^{i\theta})|$

$$\Rightarrow (M_{rad} f)(e^{i\theta}) \leq \underbrace{(M_{HL} f)}_{\text{Hardy-Littlewood}}(e^{i\theta}) = \sup_{\sigma>0} \frac{1}{2\sigma} \int_{|\theta-\varphi|<\sigma} |f(e^{i\varphi})| \frac{d\varphi}{2\pi}$$

and so $\|M_{rad} f\|_{L^2(\partial\mathbb{D}, d\theta)} \lesssim \|f\|_{H^2}$

e) For a.e. θ, $\lim_{r\to 1} f(re^{i\theta}) = f^*(e^{i\theta})$

f) $\{f^* : f \in H^2\} = \{g \in L^2(\partial\mathbb{D}) : \hat{g}_n = 0 \text{ for } n < 0\}$

3 Riesz factorization

Let $b(z,w) = \frac{|w|}{w}\left(\frac{w-z}{1-\bar{w}z}\right)$ be a Blaschke factor defined for $w \neq 0$,

then (1) for any $\{z_n\}_{n=1}^{\infty} \subseteq \mathbb{D}$ and any $z \in \mathbb{D}$, the product

$$B(z) = \prod_{j=1}^{\infty} b(z, z_j) \quad \text{converges. Moreover } \sum_{n\geq 1} (1-|z_n|) = \infty \Rightarrow B(z) \equiv 0$$

and if $\sum_{n\geq 1} (1-|z_n|) < \infty$, then $\sum \left(1 - |b(z,z_j)|\right) < \infty \Rightarrow \left(B(z) = 0 \Leftrightarrow z = z_j\right)$

(2) for $f \in N$ with zeros z_j, one has $\sum_{j\geq 1} (1-|z_j|) < \infty$

Thm 5 Let $f \in N$ and $B(z)$ the Blaschke product of zeros of f.
Then $g := f/B \in N$ with $\|g\|_N = \|f\|_N$

Thm 6 (Riesz factorization) Let $0 < p \leq \infty$, $f \in H^p$, and B the Blaschke product of zeros of f. Then, $g := f/B \in H^p$ with $\|g\|_{H^p} = \|f\|_{H^p}$.

Thm 7 Let $0 < p \leq \infty$ and $f \in H^p$. Then

a) $\|M_{rad} f\|_{L^p(\partial\mathbb{D}, d\theta)} \lesssim \|f\|_{H^p}$ $\quad (M_{rad} f = \sup_r |f(re^{i\theta})|)$

b) $\exists f^* \in L^p(\partial\mathbb{D})$ s.t. $\|f_r - f^*\|_{L^p} \xrightarrow{r \to 1} 0$, so for a.e. $f_r(e^{i\theta}) \xrightarrow{r\to 1} f^*(e^{i\theta})$

c) If $p \geq 1$, then $f_r(e^{i\theta}) = (P_r * f^*)(e^{i\theta})$

and $f(z) = i\, \mathrm{Im}\, f(0) + \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} \mathrm{Re}\, f^*(e^{i\varphi})$

d) If $p \geq 1$, then $\{f^* : f \in H^p\} = \{g \in L^p(\partial\mathbb{D}, d\theta) : \hat{g}_n = 0 \text{ for all } n < 0\}$

Thm 8 (Brother Riesz thm) Let μ be a complex Baire measure on $\partial\mathbb{D}$
s.t. $\hat{\mu}_n = 0$ for $n < 0 \Rightarrow \mu \ll d\theta$.

4 Carathéodory fcts, h^1, Herglotz representation

Def 9 If f is holomorphic on $\mathbb{D}$ with (1) $\operatorname{Re}(f) \geq 0 \quad \forall z \in \mathbb{D}$ then f is called a Carathéodory fct.
(2) $f(0) = 1$

Rem 1) $f(0)=1$ is merely a normalization condition: If any f with $\operatorname{Re}(f) \geq 0$, the $f = \alpha \tilde{f} + i\beta$ with $\alpha > 0$, $\beta \in \mathbb{R}$ and $\tilde{f}(0) = 1$.

2) By open mapping thm for holomorphic fcts and $f(0)=1$, one even has $\operatorname{Re}(f) > 0$ for all $z \in \mathbb{D}$

Thm 10 (Herglotz) If $F: \mathbb{D} \to \mathbb{C}$ is Carathéodory, then there is a probability measure μ on $\partial\mathbb{D}$ s.t. $\quad F(z) = \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \, d\mu(\theta)$

complex Poisson kernel

Remark 1) Not only is F determined by μ but also μ is determined by F since linear combinations of $\left\{ \frac{e^{i\theta}+z}{e^{i\theta}-z} \right\}_{z \in \mathbb{D}}$ and their conjugate are dense in $C(\partial\mathbb{D})$

2) Obviously, $\operatorname{Re} F(re^{i\theta}) = \int \underbrace{P_r(\theta - \varphi)}_{>0} \underbrace{d\mu(\varphi)}_{\geq 0}$

$\Rightarrow$ $\operatorname{Re} F > 0$ and $F(0) = \int d\mu = 1$, so F is Carathéodory, once μ is a prob measure

$\Rightarrow$ One-to-one correspondence between Carathéodory fcts and probability measures on $\partial\mathbb{D}$; in particular $d\mu_r(\theta) \xrightarrow[r \uparrow 1]{w} d\mu$ where $d\mu_r(\theta) = \operatorname{Re} F(re^{i\theta}) d\theta$

Thm 11 (Herglotz for harmonic fcts) Let $u \in h^1$ (harmonic and $\int |u(re^{i\theta})| \, d\theta \leq 1 \quad \forall r \in (0,1)$)
Then there is a signed measure μ on $\partial\mathbb{D}$ s.t. $u(re^{i\theta}) = \int_0^{2\pi} P_r(\theta - \varphi) \, d\mu(\varphi)$

Thm 12 If $u \in h^1$, then there is $u^* \in L^1(\partial\mathbb{D}, d\theta)$ s.t. $\lim_{r \uparrow 1} u(re^{i\theta}) = u(e^{i\theta})$ for a.e. θ
If v is harmonic conjugate of u, then there is a real-valued, measurable v^* on $\partial\mathbb{D}$ s.t. $\lim_{r \uparrow 1} v(re^{i\theta}) = v^*(e^{i\theta})$ for a.e. θ.

Rem In general, it is not true that $\|u_r - u^*\|_{L^1} \to 0$!

Thm 13 Let $f \in N$, then there is a measurable f^* on $\partial\mathbb{D}$ s.t.

$$\lim_{r \to 1} f(re^{i\theta}) = f^*(e^{i\theta}) \text{ for a.e. } \theta$$

Moreover, if $f \not\equiv 0$, $\int |\log|f^*(e^{i\theta})|| \, d\theta < \infty$, i.e., $f^* \neq 0$ a.e.

5 Boundary value measures

Thm 14 Let F be Carathéodory with probability measure μ, i.e., $F(z) = \int \frac{e^{i\theta}+z}{e^{i\theta}-z} \, d\mu(\theta)$

Then a) $\mu_{ac} = 0 \Rightarrow \mathrm{Re}\, F(re^{i\theta}) \xrightarrow{r \to 1} 0$ for a.e. θ

b) If $F^*(e^{i\theta}) = \lim_{r \to 1} F(re^{i\theta})$, then $d\mu_{ac}(\theta) = \mathrm{Re}\, F^*(e^{i\theta}) \, d\theta$

In particular $d\mu_{ac} = 0 \Leftrightarrow \mathrm{Re}\, F^*(e^{i\theta}) = 0$ for a.e. θ.

Thm 15 Let $p > 1$, F be Carathéodory with ~~a~~ probability measure μ, and $\mathrm{Re}\, F \in h^p$. Then a) $d\mu_{sing} = 0$

b) $d\mu(e^{i\theta}) = \mathrm{Re}\, F^*(e^{i\theta}) \, d\theta$ and $\mathrm{Re}\, F^* \in L^p(\partial\mathbb{D}, d\theta)$

c) $\| \mathrm{Re}\, F_r - \mathrm{Re}\, F^* \|_{L^p} \xrightarrow{r \to 1} 0$

Now extension to any h^1 or h^p fct

Thm 16 Let $u \in h^1$ and define $d\mu_r^\pm = u_\pm(re^{i\theta}) \, d\theta$ where $u_\pm = \min(0, \pm u)$.

Then there exists $r_n \to 1$ s.t. $d\mu_{r_n}^\pm \to d\mu^\pm$ and

$$u(re^{i\theta}) = \int_0^{2\pi} P_r(\theta, \varphi) \left(d\mu^+(\varphi) - d\mu^-(\varphi) \right)$$

Thm 17 Let $u \in h^1$, $u_\pm = \min(0, \pm u)$, and μ the signed measure on $\partial\mathbb{D}$ guaranteed by Herglotz (Thm 11).

If for some $p > 1$, $\sup_{0 < r < 1} \| (u_+)_r \|_{L^p} < \infty$, then $\mu_{sing} \leq 0$.

Thm 18 Let $1 < p < \infty$ and $u \in h^p$. Then $u^* \in L^p(\partial\mathbb{D})$, $u_r = P_r * u^*$, and $\|u_r - u^*\|_p \to 0$.

6 Conjugate functions

Let $Q_r(\theta) = \operatorname{Im} \frac{1+re^{i\theta}}{1-re^{i\theta}} = \frac{2r\sin\theta}{1+r^2-2r\cos\theta}$ be the conjugate Poisson kernel

Thm 19 For $p>1$, one has the $\sup_{0<r<1} \|Q_r f\|_{L^1(\partial D)} \lesssim_p \|f\|_{L^p}$.

In particular, if $f \in L^p$ is real-valued, then $(P_r + iQ_r)f \in H^1$ and so by Thm 7, ~~H~~ $(P_r+iQ_r)f$ as $L^1(\partial D)$-boundary values.

In $\mathbb{R}$ we already saw $\underbrace{\lim_{t\to 0} \langle Q_t, \varphi\rangle}_{\int_{\mathbb{R}} \frac{1}{\pi}\frac{x}{y^2+t^2} f(x-y)\,dy} = H(\varphi) = \lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} \frac{\varphi(y)}{x-y}\,dy$ Hilbert transform

Similarly on ∂D we ~~defi~~ make define its Hilbert transform $\tilde{f}$ on ∂D by

Def If $f \in L^1(\partial D, d\theta)$ is real-valued, then

$$\tilde{f}(e^{i\theta}) = \lim_{r\nearrow 1} (Q_r * f)(e^{i\theta}) = \lim_{\epsilon\to 0} \int_{|\varphi-\theta|>\epsilon} Q_1(\theta-\varphi) f(e^{i\varphi})\,d\varphi$$

$\tilde{f}$ is also called the conjugate to f.

Thm 20 If $f \in L^1(\partial D)$ real-valued and $\tilde{f} \in L^1$, then

a) $(P_r + iQ_r) f \in H^1$

b) $Q_r f = P_r \tilde{f}$

c) $\|Q_r f - \tilde{f}\|_{L^1} \to 0$.

In particular, if $f \in L^p$ for some $p>1$, then a) - c) hold

Corollary 21 If $\tilde{f} \in L^1$ (in particular, if $f \in L^p$, some $p>1$), then

$$(\tilde{f})^\wedge(n) = -i\,\mathrm{sgn}(n)\,\hat{f}(n).$$

Thm 22 (Conjugate fct duality)

Let $1<p<\infty$. a) $\sup_{0<r<1} \|Q_r f\|_p \lesssim_p \|f_p\| \ \forall f \Rightarrow \sup_{0<r<1} \|Q_r f\|_{p'} \lesssim \|f\|_{p'} \ \forall f$

b) $\|\tilde{f}\|_p \lesssim \|f\|_p \ \forall f \Rightarrow \|\tilde{f}\|_{p'} \lesssim \|f\|_{p'}$.

Let $(H_r f)(e^{i\theta}) := \int_{|\varphi-\theta|>1-r} \cot\left(\frac{\theta-\varphi}{2}\right) f(e^{i\varphi})\, d\varphi$

Thm 23 For all $r\in(0,1)$ and $f\in L^1$, one has
$$|((Q_r - H_r) f)(e^{i\theta})| \le (\pi+2\pi)(M_{HL} f)(e^{i\theta})$$

Prop 24 Let $f\in C^1(\partial\mathbb{D})$, then $(Q_r f)(e^{i\theta}), (H_r f)(e^{i\theta}) \xrightarrow{r\nearrow 1} \tilde{f}(e^{i\theta})$ uniformly.

Thm 25 Let $f\in L^1(\partial\mathbb{D})$, then $(H_r f)(e^{i\theta}) \xrightarrow{r\nearrow 1} \tilde{f}(e^{i\theta})$ for a.e. θ.

Thm 26 Let $f\in L^1(\partial\mathbb{D})$ real-valued, then $\sup_{0<r<1} \|Q_r\|_{L^1\to L^{1,\infty}} \le 4$, i.e.,
$$|\{\theta : |(Q_r f)(e^{i\theta})|>t\}| \le \frac{4\|f\|_1}{t}$$
Moreover, $|\{\theta: |\tilde{f}(e^{i\theta})|>t\}| \le \frac{4\|f\|_1}{t}$.

If $f\in L^p(\partial\mathbb{D})$, then $\|\tilde{f}\|_{L^p} \lesssim_p \|f\|_{L^p}$. (Hilbert transform L^p-bdd + weak-type (1,1))

Let $(M_{conj} f)(e^{i\theta}) = \sup_{0<r<1} |(Q_r f)(e^{i\theta})|$ (conjugate fct)

$(M_{HT} f)(e^{i\theta}) = \sup_{0<r<1} |(H_r f)(e^{i\theta})|$ (Hilbert transf.)

$(M_a f)(e^{i\theta}) = \sup_{0<r<1} |((H_r - Q_r) f)(e^{i\theta})|$

Thm 27 For $f\in L^p$, $p>1$: $(M_{conj} f)(e^{i\theta}) \le (\underset{\substack{\uparrow \\ \text{Hardy-Littlewood}}}{M_{HL}} \tilde{f})(e^{i\theta})$

$$(M_{HT} f)(e^{i\theta}) \le (M_{HL}\tilde{f})(e^{i\theta}) + (\pi+2\pi)(M_{HL} f)(e^{i\theta})$$

In particular, $M_{conj} f, M_{HT} f \in L^p(\partial\mathbb{D})$.

Thm 28 Let $1<p<\infty$ and $f\in L^p \Rightarrow \|Q_r f - \tilde{f}\|_p \xrightarrow{r\nearrow 1} 0$
$$\|H_r f - \tilde{f}\|_p \xrightarrow{r\nearrow 1} 0$$

9 Boundary values of analytic fcts in upper half plane

Idea: use Möbius transform (bijective)

Note φ sets up a bijection between $H^p(\mathbb{C}_+)$ and $H^p(\mathbb{D})$ only for $p=\infty$!

$\mathbb{C}_+ \to \mathbb{D}$

$z \mapsto \varphi(z) = \frac{i-z}{i+z}$

$(\varphi(\infty) = -1)$

$\mathbb{D} \to \mathbb{C}_+$

$w \mapsto \psi(w) = \varphi^{-1}(w) = i\,\frac{1-w}{1+w}$; $\psi: \partial\mathbb{D} \to \mathbb{R}\cup\{\infty\}$

$(\psi(-1) = \infty)$

to transfer previous results on $H^p(\mathbb{D})$, $N(\mathbb{D})$, and $h^p(\mathbb{D})$ to $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$.

Recall that Carathéodory fcts were analytic fcts on $\mathbb{D}$ with ~~Fr~~ $\operatorname{Re} F \geq 0$ and normalization condition $F(0) = 1$. Moreover, there is a unique positive prob. measure μ on $\partial\mathbb{D}$ s.t.

$$F(w) = \int_0^{2\pi} \frac{e^{i\theta}+w}{e^{i\theta}-w}\, d\mu(\theta) \ , \quad \int_0^{2\pi} d\mu(\theta) = 1 = \mu(\partial\mathbb{D}), \quad F(0) = 1$$

If normalization condition $F(0)=1$ is dropped, then μ will no longer be normalized and we need an additional $+ib$ term if $F(0) = b$.

Def Herglotz- / Nevanlinna- / Pick fct is a fct ~~$f: \mathbb{C}_+ \to f(z)$~~ that is analytic in $\mathbb{C}_+$ and maps $\mathbb{C}_+$ into itself, i.e., ~~$\operatorname{Im} z > 0 \Rightarrow f(z)$~~

$\operatorname{Im} z > 0 \Rightarrow \operatorname{Im} f(z) > 0$.

$\Rightarrow$ If f is Pick on $\mathbb{C}_+$, then $F(w) := -i f(\psi(w))$ is "weakly Herglotz" on $\mathbb{D}$, i.e. satisfies $\operatorname{Re} F > 0$. Moreover $f(z) = i\,F(\varphi(z))$ has then the form

$$f(z) = b + i\int \frac{e^{i\theta} + \varphi(z)}{e^{i\theta} - \varphi(z)}\, d\mu(\theta) \qquad \text{write } e^{i\theta} = \varphi(x)$$

$$\to f(z) = az + b + \int \frac{1+xz}{x-z}\, d\mu(x) \ , \quad a = \lim_{y_0\to\infty} \frac{\operatorname{Im} f(iy_0)}{y_0}$$

$$= az + b + \int \left(\frac{1}{x-z} \cancel{a} - \frac{x}{1+x^2}\right) d\rho(x) \quad \text{for } d\rho(x) = (1+x^2)\,d\mu; \ \cancel{\int d\rho < \infty}$$

with $\int \frac{d\rho}{1+x^2} < \infty$ (since $\int d\mu < \infty$)

If $\int d\rho < \infty \rightsquigarrow f(z) = az + c + \int \frac{d\rho}{x-z}$

~~$\Rightarrow$ Thm 29 (Herglotz represen~~

Thm 29 (Herglotz representation)

Let $f: \mathbb{C}_+ \to \mathbb{C}_+$ be Pick, then it has the representation

$$f(z) = a z + b + \int_{\mathbb{R}} d\mu(x) \, \frac{1+xz}{x-z} \quad \text{where } a \geq 0,\ b \in \mathbb{R},\ b = \operatorname{Re}(f(i)),$$

and $\mu \geq 0$ is a finite measure on $\mathbb{R}$

Equivalently, f has form $f(z) = az + b + \int_{\mathbb{R}} d\rho(x) \left(\frac{1}{x-z} - \frac{x}{1+x^2} \right)$ where $\rho \geq 0$ is a measure obeying $\int \frac{d\rho}{1+x^2} < \infty$. If additionally $\int d\rho < \infty$, then also $f(z) = az + \underset{b - \int d\rho \frac{x}{1+x^2}}{\underset{\parallel}{c}} + \int \frac{d\rho(x)}{x-z}$

Moreover, a) $\operatorname{Im} f(x_0 + iy_0) = a y_0 + \int \frac{d\rho(x)}{(x-x_0)^2 + y_0^2}$

b) f has non-tangential boundary values on $\mathbb{R}$

c) $a = \lim_{y_0 \to \infty} \frac{\operatorname{Im} f(iy_0)}{y_0}$

d) If $d\rho(x) = w(x) dx + d\rho_{sing}$, then $w(x) = \frac{1}{\pi} \operatorname{Im} f(x + i0)$

e) $d\rho(\{x_0\}) = \lim_{\epsilon \downarrow 0} \epsilon \operatorname{Im} f(x_0 + i\epsilon)$

f) $d\rho_{sing}$ is supported on $\{x : \operatorname{Im} f(x+i\epsilon) \xrightarrow{\epsilon \to 0} \infty\}$

g) $\frac{1}{\pi} \operatorname{Im} f(x+i\epsilon)\, dx \xrightarrow[\epsilon \to 0]{*} d\rho$ when integrated against $C_c(\mathbb{R})$-fcts.

We will shortly see that $\varphi: \mathbb{C}_+ \to \mathbb{D}$ sets up a bijection between $H^p(\mathbb{C}_+)$ and $H^p(\mathbb{D})$ only for $p = \infty$, but generally not for $0 < p < \infty$.

Thm 30 Let $u: \mathbb{C}_+ \to \mathbb{C}_+$ be bounded and harmonic. Then $u^*(x) := \lim_{\epsilon \to 0} u(x + i\epsilon)$ exists for Lebesgue all $x \in \mathbb{R}$ ($L^\infty(\mathbb{R})$ on left, $L^\infty(\mathbb{C}_+)$ on right)

with $\|u^*\|_\infty = \|u\|_\infty$

and for all $z_0 = x_0 + iy_0 \in \mathbb{C}_+$,

$$u(z_0) = \frac{1}{\pi} \int \frac{y_0}{(x-x_0)^2 + y_0^2} u^*(x)\, dx \quad (\text{cf. Thm } 11)$$

If f is holomorphic on $\mathbb{C}_+$ with $\operatorname{Im} f \in L^\infty(\mathbb{C}_+)$, then $f^*(x) := \lim_{\epsilon \to 0} f(x + i\epsilon)$ exists for Lebesgue-all $x \in \mathbb{R}$ and $\|\operatorname{Im} f^*\|_\infty = \|\operatorname{Im} f\|_\infty$.

Moreover, for an $\alpha \in \mathbb{R}$, $\quad f(z) = \alpha + \frac{1}{\pi} \int_{\mathbb{R}} \frac{\operatorname{Im} f^*(x)}{x-z} dx.$

If $f^* \in L^1(\mathbb{R}, dx)$, then $\alpha = \lim_{y \to \infty} f(x_0 + iy)$ for any $x_0 \in \mathbb{R}$.

For $0<p\le\infty$ one may define $H^p(\mathbb{C}_+)$ as those fcts f holomorphic in $\mathbb{C}_+$,
with $\|f\|_p^p = \sup_{0<y<\infty} \int |f(x+iy)|^p dx$ resp $\|f\|_\infty = \sup_{0<y<\infty} \sup_{x\in\mathbb{R}} |f(x+iy)|$
(analogously h^p being real-valued harmonic fcts with finite $\|\cdot\|_p$-norm)

Thm 31 Let $0<p<\infty$, $f\in H^p(\mathbb{C}_+)$. Then $|f(z)| \le \left(\frac{2}{\pi \operatorname{Im}(z)}\right)^{1/p} \|f\|_p$

Cor 32 Let $u\in h^p(\mathbb{C}_+)$, $0<p<\infty$ $\Rightarrow$ there is at most one real-valued fct. v on $\mathbb{C}_+$ s.t. $u+iv\in H^p(\mathbb{C}_+)$

Thm 33 ($\sim$ Paley-Wiener)

a) Let $g\in L^2([0,\infty), dk)$, then for $z\in\mathbb{C}_+$, the integral $f(z)=\int_{\mathbb{R}} \frac{dk}{\sqrt{2\pi}} g(k) e^{ikz}$ converges absolutely and defines an element in H^2 with $\|f\|_{H^2} = \|g\|_{L^2}$.

b) Conversely, if $f\in H^2(\mathbb{C}_+)$ and $g_y(k) = \frac{1}{\sqrt{2\pi}} \int dx\, f(x+iy) e^{-ikx}$, then for $0<y<w$, one has $e^{ky} g_y(k) = e^{kw} g_w(k)$
Moreover, each $g_w(k)$ is supported on $[0,\infty)$ and there is $g\in L^2([0,\infty), dx)$ s.t. $g_y(k) = e^{-ky} g(k)$ and f that we started with can be written as
$f(z) = \int_{\mathbb{R}} \frac{dk}{\sqrt{2\pi}} g(k) e^{ikz}$ with this g.

For c)-g) let $f\in H^2(\mathbb{C}_+)$ and f and g are related by $f(z)=\int dk\, e^{ikz} g(k)$

c) If $f^*(x) := (\mathcal{F}^{-1} g)(x)$ (inverse FT in L^2), then
$f(\cdot+iy) \xrightarrow[y\to 0]{} f^*$ in $L^2(\mathbb{R}, dx)$

d) For $x_0+iy_0\in\mathbb{C}_+$, we have $f(x_0+iy_0) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y_0}{(x-x_0)^2+y_0^2} f^*(x)\,dx$

e) For $z_0\in\mathbb{C}_+$, we have $f(z) = \frac{1}{\pi i}\int \frac{1}{x-z} \operatorname{Re} f^*(x)\,dx$

f) $\sup_{0<y<\infty} |f(x+iy)| \le (M_{HL} f^*)(x) = \sup_R \frac{1}{2R} \int_{|x-y|<R} |f^*(y)|\,dy$

g) $\lim_{y\searrow 0} f(x+iy) = f^*(x)$ for a.e. $x\in\mathbb{R}$

$\rightarrow$ analogous Thm for real-valued harmonic fcts in $h^2(\mathbb{C}_+)$
(Thm 5.9.6)

Now for the development of general $H^p(\mathbb{C}_+)$ theory.

Prop 34 There exists an $H^\infty(\mathbb{C}_+)$ fct vanishing exactly at $\{z_j\}_{j\in\mathbb{N}}$ (counting multiplicity) iff $\sum_{j\geq 1} \frac{\operatorname{Im} z_j}{1+|z_j|^2} < \infty$ and if that holds, then $\prod_{j\geq 1} b_{z_j}(z_0)$ is such a fct.

Blaschke factor

! p 507 in Simon. Most importantly: One can prove a Riesz factorization thm and that fcts in $H^p(\mathbb{C}_+)$ also have pointwise boundary values.

Finally, consider Hilbert transform of measures

$H_\mu(x) = \lim_{\epsilon\to 0} \frac{1}{\pi} \int_{\mathbb{R}} \operatorname{Re}\left(\frac{1}{t-x-i\epsilon}\right) d\mu(t)$ which exists for a.e. x since $f(z) = \int \frac{d\mu(t)}{t-z}$ is holomorphic in $\mathbb{C}_+$ with $\operatorname{Im} f > 0$, so $g(z) \equiv e^{if(z)} \in H^\infty(\mathbb{C}_+)$, i.e., g has boundary values which are a.e. non-zero, which means that $f = i\log(g)$ does have a limit.

↳ recall Lemma 1, Thm 2,

Let $(H_\epsilon \mu)(x) = \frac{1}{\pi} \int_{|t-x|>\epsilon} \frac{d\mu(t)}{t-x}$ and $(Q_\epsilon \mu)(x) = \frac{1}{\pi} \int_{\mathbb{R}} \left(\operatorname{Re} \frac{1}{t-x-i\epsilon}\right) d\mu(t)$

Thm 35 For any measure $\mu \geq 0$, we have ~~$|(H_\epsilon \mu)(x) -$~~

$$|(H_\epsilon \mu)(x) - Q_\epsilon \mu)(x)| \leq \left(1+\frac{2}{\pi}\right)(M_{HL}\,\mu)(x)$$

In particular, $(H\mu)(x) = \lim_{\epsilon\to 0} (H_\epsilon \mu)(x)$ for a.e. $x \in \mathbb{R}$.

Thm 36 (Boole's equality) Let μ be a finite (positive) point measure with a finite number of pure points.

$$\Rightarrow |\{x : \pm(H_\mu)(x) > \lambda\}| = \frac{\|\mu\|}{\pi\lambda}$$

not only "$\leq$" !

In fact, this holds for any purely singular (positive) finite measure μ (i.e., $\mu(\mathbb{R}\setminus E) = 0$ for any $\operatorname{Leb}(E) = 0$)

Thm 37 Let μ be a finite (positive) measure on $\mathbb{R}$ with $F_\mu(z) = \int \frac{d\mu(t)}{t-z}$ and $(H_\mu)(x) = \frac{1}{\pi} \operatorname{Re} F_\mu(x+i0) \Rightarrow |\{x : |F_\mu(x+i0)| \geq \lambda\}| \leq \frac{4\|\mu\|}{\lambda}$

$$|\{x : |(H\mu)(x)| \geq \lambda\}| \leq \frac{4\|\mu\|}{\pi\cdot\lambda}$$

Some more remarks on H^∞ (Jaksic p. 25-29)

Thm 38 Let $f \in H^\infty(\mathbb{C}_+)$, then $\lim_{\epsilon\downarrow 0} f(x+i\epsilon) = f^*$ exists for Lebesgue-all $x \in \mathbb{R}$

and $f^* \in L^\infty(\mathbb{R})$

Thm 39 Let $f \not\equiv 0$, $f \in H^\infty(\mathbb{C}_+)$ and $f^* = \lim_{\epsilon\downarrow 0} f(x+i\epsilon)$. Then

$\int \frac{|\log|f(x)||}{1+x^2} dx < \infty$, in particular $f \neq 0$ Lebesgue-surely.

In fact, if $\alpha \in \mathbb{C}$, then either $f(z) \equiv \alpha$ or the set $\{x \in \mathbb{R} : f(x) = \alpha\}$ has Lebesgue measure zero.

Thm 40 Let $f : \mathbb{C}_+ \to \mathbb{C}_+$ be holomorphic, then

~~a) $F(x) := \lim$~~

a) $f^*(x) := \lim_{\epsilon\downarrow 0} f(x+i\epsilon)$ exists and is finite for Lebesgue-almost ~~every~~ every x.

b) If $\alpha \in \mathbb{C}$, then either $f(z) \equiv \alpha$ or $|\{x : f(x) = \alpha\}| = 0$

Thm 41 Let μ be a finite or complex measure. Then for $F_\mu(z) = \int \frac{d\mu(t)}{t-z}$:

a) $F_\mu(x) = \lim_{\epsilon\downarrow 0} F_\mu(x+i\epsilon)$ exists and is finite a.e

b) If $F_\mu \not\equiv 0$, then $\int \frac{|\log|F_\mu(x)||}{1+x^2} dx < \infty$ (i.e. $F_\mu(x) \neq 0$ Lebesgue-surely)

c) If $F_\mu \not\equiv 0$, then for any $\alpha \in \mathbb{C}$, $|\{x \in \mathbb{R} : F_\mu(x)^q = \alpha\}| = 0$.

Thm 42 Let $\underset{= f\mu + \nu_s}{\nu}$ be a complex measure and μ be a positive measure.

$\Rightarrow \lim_{\epsilon\downarrow 0} \frac{F_\nu(x+i\epsilon)}{F_\mu(x+i\epsilon)} = f(x)$ for μ_{sing}-almost every x

(Poltoratski)

<u>Thm 43</u> Let $\underset{\sim}{\nu}$ be a complex measure and μ be a positive measure.
$= f\mu + \nu_s$ Let $P_\mu = \operatorname{Im} F_\mu$

$\Rightarrow$ a) $\lim\limits_{\epsilon \to 0} \dfrac{P_\nu(x+i\epsilon)}{P_\mu(x+i\epsilon)} = f(x)$ for μ-a.e. x

In particular, $\nu \perp \mu \Leftrightarrow \lim\limits_{\epsilon \to 0} \dfrac{P_\nu(x+i\epsilon)}{P_\mu(x+i\epsilon)} = 0$.

b) If additionally $\nu \geq 0$, then $\lim\limits_{\epsilon \to 0} \dfrac{P_\nu(x+i\epsilon)}{P_\mu(x+i\epsilon)} = \infty$ for ν_s-a.e. x