Winter term
2020/2021

Topics in PDE
Homework Sheet 6

Exercise 6.1
Let C_3 be the ternary Cantor set discussed in the lecture. Prove that C_3

1. is uncountable
2. contains no intervals
3. is compact
4. is perfect (i.e., closed and has no isolated points, respectively every point in C_3 is an accumulation point)
5. is nowhere dense.

Let $\lambda = \sum_{n=1}^{\infty} a_{n,\lambda} 3^{-n}$ with $a_{n,\lambda} \in \{0, 1, 2\}$ and

$$F(\lambda) := \frac{1}{2^{N_\lambda}} + \frac{1}{2} \sum_{n=1}^{N_\lambda - 1} \frac{a_{n,\lambda}}{2^n} = \begin{cases} \sum_{n=1}^{\infty} \frac{a_n}{2^n}\,, & \lambda = \sum_{n=1}^{\infty} \frac{2a_n}{3^n} \in C_3\,,\ a_n \in \{0,1\} \\ \sup_{y \le \lambda,\, \lambda \in C_3} F(y)\,, & \lambda \in [0,1] \setminus C_3 \end{cases}$$

be the Cantor function with N_λ being the smallest n such that $a_{n,\lambda} = 1$ (if it exists) and ∞ if all $a_{n,\lambda} \in \{0, 2\}$. Show that F is Hölder continuous with exponent $\log 2/\log 3$.

Exercise 6.2
Let μ be a positive measure on $\mathbb{R}$, $F_\mu(z) = \int_{\mathbb{R}} (t - z)^{-1} d\mu(t)$ (Borel transform), and $P_\mu(z) = \operatorname{Im} F_\mu(z)$ (Poisson transform). Show that for all $x \in \mathbb{R}$ and $y > 0$,

$$\frac{P_\mu(x + iy)}{y} = \int_0^{1/y^2} \mu(I(x, \sqrt{u^{-1} - y^2}))\, du$$

with the abbreviation $I(x, r) := (x - r, x + r)$ for the interval of length $2r$ centered around x.

Exercise 6.3
Suppose μ is a measure on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} (1 + |x|)^{-1} d\mu(x) < \infty$ and let $G_\mu(y) = \int_{\mathbb{R}} (x - y)^{-2} d\mu(x)$. Prove that $\{y \in \mathbb{R} : G(y) = \infty\}$ is dense in a support of μ. (Hints: Recall that $\lim_{\varepsilon \searrow 0} F_\mu(E + i\varepsilon) \equiv \lim_{\varepsilon \searrow} \int (x - (E + i\varepsilon))^{-1}\, d\mu(x)$ exists and is finite for Lebesgue a.e. E and that μ_{ac} is supported on $\{E \in \mathbb{R} : \operatorname{Im} F_\mu(E + i0) > 0\}$ and that μ_{sing} is supported on $\{E \in \mathbb{R} : \operatorname{Im} F_\mu(E + i0) = \infty\}$.)

Exercise 6.4
Prove (the baby version of) Jensen's formula: let $U(z)$ be holomorphic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with $U(0) \neq 0$. Let $r \in (0, 1)$ and assume that U has no zeros on $|z| = r$. If $\{\alpha_j\}_{j=1}^n$ denote the (finitely many) zeros of U in $|z| < r$ counting multiplicities, then

$$|U(0)| \prod_{j=1}^{n} \frac{r}{|\alpha_j|} = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |U(re^{it})|\, dt \right) . \tag{1}$$

(Hint: consider $V(z) := U(z)\prod_{j=1}^{n} B(z,\alpha_j,r)^{-1}$ where $B(z,\alpha,r) = \frac{r(\alpha-z)}{r^2-\overline{\alpha}z}$ is a Blaschke factor. Recall also that holomorphic functions $\psi : \Omega \to \mathbb{C}$ defined in simply connected subsets $\Omega \subseteq \mathbb{C}$ which have no zeros on Ω, have the representation $\psi = \exp(\varphi)$ for some holomorphic $\varphi : \Omega \to \mathbb{C}$. Finally recall the mean-value property of harmonic functions.)

Remark: In fact, α_j are actually allowed to be zeros on $|z| = r$ as well. Moreover, by a simple affine transformation, (1) holds for U defined in $\{z \in \mathbb{C} : |z - z_0| < R\}$ for any $z_0 \in \mathbb{C}$, $R > 0$, whenever $U(z_0) \neq 0$. Finally, if U is meromorphic with poles $\{\beta_\ell\}_{\ell=1}^{m}$, (1) holds when the left side is divided by $\prod_{\ell=1}^{m} \frac{r}{|\beta_j|}$ by the same "Blaschke trick".