Winter term
2020/2021

Topics in PDE
Homework Sheet 5

Exercise 5.1
Compute the Fourier transform of

a) $h^{d/2}\mathrm{e}^{-h|x|^2/2}$ in $\mathbb{R}^d$ for $h>0$ and

b) $|\xi|^{-\alpha}$ in $\mathbb{R}^d$ for $0<\alpha<d$ in the sense of tempered distributions, i.e., compute $\mathcal{F}[|\cdot|^{-\alpha}f](x)$ for $f\in\mathcal{S}$.

Exercise 5.2

1. Let $m\in L^\infty(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. Show that
$$(m(-i\nabla)f)(x) := \mathcal{F}^{-1}[m\hat{f}](x) = \int_{\mathbb{R}^d}\check{m}(x-y)f(y)\,dy$$
for all $f\in L^2(\mathbb{R}^d)$. In particular, the integral converges for all $x\in\mathbb{R}^d$. (Hint: Recall that $\mathcal{S}$ is dense L^2 and use Plancherel.)

2. Let $t\in\mathbb{R}$ and $\varepsilon>0$. Show that
$$\mathcal{F}[\mathrm{e}^{-i\pi(t-i\varepsilon)\xi^2}](x) = \frac{\mathrm{e}^{-\frac{|x|^2}{\varepsilon+it}}}{(\varepsilon+it)^{d/2}}\,,\quad x\in\mathbb{R}^d\,.$$
Next, for $f\in L^2(\mathbb{R}^d)$ and $\check{f}=\mathcal{F}^{-1}[f]$ show that
$$\mathcal{F}[\mathrm{e}^{-i\pi t\xi^2}\check{f}](x) = \lim_{R\to\infty}\int_{|y|\le R}\frac{\mathrm{e}^{-\frac{|x-y|^2}{it}}}{(it)^{d/2}}f(y)\,dy$$
where the limit is understood in $L^2(\mathbb{R}^d)$ and $(it)^{d/2}\equiv((it)^{1/2})^d$ where $\operatorname{Re}z^{1/2}>0$ for $z\in\mathbb{C}\setminus(-\infty,0]$.

3. Let $\kappa\in\mathbb{C}$ with $\operatorname{Re}\kappa>0$ and $f\in L^2(\mathbb{R}^3)$. Invoke part 1. to show that
$$[(-\Delta+\kappa^2)^{-1}f](x) = \pi\int_{\mathbb{R}^3}\frac{\mathrm{e}^{-2\pi\kappa|x-y|}}{|x-y|}f(y)\,dy\,.$$

Exercise 5.3
Let $\mathcal{H}=L^2(0,1)$, $A:\mathcal{D}(A)\to\mathcal{H}$ with $A=-d_x^2$ and $\mathcal{D}(A)=\{f\in H^2[0,1]:\ f(0)=f(1)=0\}$. Let
$$q(f)=|f(c)|^2\,,\qquad f\in H^1[0,1]\,,\qquad c\in(0,1)$$
be the form associated to the Dirac distribution centered at c. Show that for $f\in\mathcal{Q}(A)=\{f\in H^1[0,1]:\ f(0)=f(1)=0\}$, q is relatively form bounded with respect to q_A with arbitrary small form bound.

Exercise 5.4 (Rollnik potentials)

Definition 0.1 (Rollnik potential). *A measurable function* $V : \mathbb{R}^3 \to \mathbb{R}$ *is called* Rollnik potential, *whenever*

$$\|V\|_R^2 := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(x)||V(y)|}{|x-y|^2} \, dx \, dy < \infty \, .$$

The set of Rollnik potentials is denoted by R*.*

Show the following assertions.

1. Let $V \in L^1 \cap L^2$. Then $V \in R$ with $\|V\|_R \leq \sqrt{3}(2\pi)^{1/3} \|V\|_1^{1/3} \|V\|_2^{2/3}$.
2. Let $V \in R + L^\infty$ be compactly supported (i.e., there is $\rho > 0$ such that $V(x) = 0$ for all $|x| > \rho$). Then $V \in R \cap L^1$. In particular, any $V \in R + L^\infty$ is locally integrable. (Hint: Use the Hardy–Littlewood–Sobolev inequality.)
3. Let $V \in L^1 \cap L^2$. Then, for any $a > 0$, one has
$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{V(x)V(y)}{a^2 + |x-y|^2} \, dx \, dy = \pi \int_{\mathbb{R}^3} \frac{|\hat{V}(\xi)|^2}{|\xi|} \mathrm{e}^{-2\pi a |\xi|} \, d\xi \, .$$
4. Let $0 \leq V \in L^1(\mathbb{R}^3)$. Then $V \in R$ if and only if
$$\int \frac{|\hat{V}(\xi)|^2}{|\xi|} \, d\xi < \infty \, .$$
If $V \in R \cap L^1$, then
$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{V(x)V(y)}{|x-y|^2} \, dx \, dy = \pi \int_{\mathbb{R}^3} \frac{|\hat{V}(\xi)|^2}{|\xi|} \, d\xi$$
even if V is not assumed to be positive.
5. Let $V \in R$. Then
$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{V(x)V(y)}{|x-y|^2} \, dx \, dy \geq 0 \, .$$
(Hint: Use the previous part of the exercise.)
6. Let $V \in R$. Then for any $a > 0$ there is $b = b(a) > 0$ such that
$$\langle \psi, |V| \psi \rangle \leq a \langle \psi, -\Delta \psi \rangle + b \|\psi\|^2 \, , \quad \psi \in C_c^\infty(\mathbb{R}^3) \, .$$