Topics in PDE
Homework Sheet 4

Exercise 4.1

1. Show that $(-\Delta+1)^{-1}$ is not compact on $L^2(\mathbb{R}^d)$, i.e., find a bounded sequence $(u_n)_{n\in\mathbb{N}}$ in $L^2(\mathbb{R}^d)$ such that $((-\Delta+1)^{-1}u_n)_{n\in\mathbb{N}}$ has no convergent subsequence in $L^2(\mathbb{R}^d)$.

2. Let $B_x(R)=\{y\in\mathbb{R}^d : |x-y|\leq R\}$. For which $\alpha\geq 0$, $d\in\mathbb{N}$, and $1\leq p\leq\infty$ does $B_0(1)\to\mathbb{R} : x\mapsto |x|^{-\alpha}$ belong to $W^{1,p}(B_0(1))$?

   (Remark: If $\{r_k\}_{k\in\mathbb{N}}$ is a countable, dense subset of $B_0(1)$ and $u(x)=\sum_{k=1}^{\infty}2^{-k}|x-r_k|^{-\alpha}$ for $x\in B_0(1)$, then the example shows $u\in W^{1,p}(B_0(1))$ but is yet unbounded on each open subset of $B_0(1)$.)

Exercise 4.2 (Uncertainty principle I)
Let $R>0$ and $f\in C_c^\infty(\mathbb{R})$ be such that $\operatorname{supp} f\subseteq \overline{B}_0(R)\equiv\{x\in\mathbb{R} : |x|\leq R\}$. Show that $\hat{f}$ is holomorphic and satisfies $|\hat{f}(\xi)|\leq \mathrm{e}^{2\pi R|\operatorname{Im}\xi|}\|f\|_1$ for $\xi\in\mathbb{C}$. Conclude that $\operatorname{supp}\mathcal{F}[f]$ cannot be compact, unless $\hat{f}\equiv 0$.

Exercise 4.3 (Uncertainty principle II)
Let $\mathcal{H}$ be a complex Hilbert space with $\dim\mathcal{H}\geq 1$. Let P,Q be two self-adjoint operators on $\mathcal{H}$ such that

$$(\varphi,[P,Q]\varphi) := (P\varphi,Q\varphi)-(Q\varphi,P\varphi)=\frac{1}{i}\|\varphi\|^2$$

for all $\varphi\in\mathcal{D}(P)\cap\mathcal{D}(Q)$. Show the following assertions.

1. One necessarily has $\dim\mathcal{H}=\infty$. (Hint: Think of the trace of a commutator in the finite-dimensional setting.)

2. Assume that P and Q were bounded. Show that neither P nor Q can have an eigenvalue.

3. Show that neither P nor Q can be bounded. (Hint: Every self-adjoint operator has non-empty spectrum, see Exercise 2.4. Recall also Lemma 1.2.8.)

Exercise 4.4 (Uncertainty principle III / Hardy's inequality)
Let $3\leq d\in\mathbb{N}$, $g\in C_c^\infty(\mathbb{R}^d)$, and $f(x)=\langle x\rangle_\varepsilon^{-\alpha}\, g(x)$ where $\langle x\rangle_\varepsilon := (|x|^2+\varepsilon^2)^{1/2}$. Show that

$$\int_{\mathbb{R}^d}|\nabla f(x)|^2\,dx\geq(-\alpha^2+\alpha(d-2))\int_{\mathbb{R}^d}\frac{|f(x)|^2}{|x|^2}\,dx$$

and show that the optimal value of α is given by $(d-2)/2$. (Hint: compute first $\|\nabla f\|_2^2$ and let then $\varepsilon \searrow 0$.)
Next, show that for $\psi \in H_0^1(\mathbb{R}_+)$, one has

$$\int_0^\infty |\psi'(r)|^2\, dr - \frac{1}{4}\int_0^\infty \frac{|\psi(r)|^2}{r^2}\, dr \geq 0\,.$$

(Hint: you may use that $C_c^\infty(\mathbb{R}_+)$ is dense in $H_0^1(\mathbb{R}_+)$.) Prove that there is no non-vanishing $\psi \in H_0^1(\mathbb{R}_+)$ that saturates the inequality, i.e., there is no ψ such that the left side identically vanishes. Moreover, show that the value $1/4$ is optimal in the sense that an inequality where $1/4$ is replaced by $1/4+\varepsilon$ for some $\varepsilon > 0$ is generally false. (Recall that $H_0^s(\Omega) = \overline{C_c^\infty(\Omega)}$ for some $\Omega \subseteq \mathbb{R}^d$ where the closure is with respect to the H^s norm. In this case, this simplifies to $H_0^1(\mathbb{R}_+) = \{f \in H^1(\mathbb{R}_+) :\ f^{(j)}(0) = 0\,, j = 0, 1\}$.)