Winter term
2020/2021

Topics in PDE
Homework Sheet 3

Exercise 3.1
Let $V : \mathbb{R}^d \to \mathbb{C}$ be measurable and $M_V : \mathcal{D}(M_V) \to L^2(\mathbb{R}^d)$ be the multiplication operator that acts as $(M_V\psi)(x) := V(x)\psi(x)$ for $\psi \in \mathcal{D}(M_V)$ and $x \in \mathbb{R}^d$. Show the following assertions.

a) M_V is closed on its maximal domain $\mathcal{D}(M_V) = \{\psi \in L^2(\mathbb{R}^d) : V\psi \in L^2(\mathbb{R}^d)\}$.

b) The spectrum is the essential range, i.e.,

$$\sigma(M_V) = \operatorname{ess\,ran}(V) := \{z \in \mathbb{C} : \forall\varepsilon > 0 : |\{x \in \mathbb{R}^d : |V(x) - z| < \varepsilon\}| > 0\}$$

where $|A|$ denotes the Lebesgue measure of a subset $A \subseteq \mathbb{R}^d$.

c) One has $M_V \in \mathfrak{B}(L^2(\mathbb{R}^d))$ if and only if $V \in L^\infty$ and in this case $\|M_V\| = \|V\|_\infty$.

d) Let $\Omega \subseteq \mathbb{R}^d$ be open and $V : \Omega \to \mathbb{C}$ be continuous. Then $C_c(\Omega)$ is a core for M_V.

e) Let $V : \mathbb{R}^d \to \mathbb{R}$ be continuous, $\varphi \in \mathcal{D}(M_V)$, and $b \in \mathbb{R}$. Show that

$$\begin{aligned}&\left(\varphi, \lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0} \frac{1}{2\pi i}\int_{-\infty}^{b+\delta} ((M_V - \lambda - i\varepsilon)^{-1}\varphi - (M_V - \lambda + i\varepsilon)^{-1}\varphi)\, d\lambda\right) \\ &= \int_{\mathbb{R}^d} |\varphi(x)|^2 \mathbf{1}_{\{x\in\mathbb{R}^d :\, V(x)\le b\}}\, dx\,.\end{aligned}$$

Exercise 3.2
Let V be a real-valued function and M_V be the above multiplication operator defined on its maximal domain. Show that M_V is self-adjoint if this domain is dense in L^2.

Exercise 3.3
Let $V(x) = x$ and consider the multiplication operator M_0 in $L^2(\mathbb{R})$ defined on $\mathcal{D}(M_0) = \{f \in \mathcal{D}(M_V) : \int_{\mathbb{R}} f(x)\, dx = 0\}$ (Here $\mathcal{D}(M_V)$ is the maximal domain as in Exercise 3.1.) Show that M_0 is closed, i.e., in particular, $\mathcal{D}(M_0)$ is not a core for M_V.

Exercise 3.4
Let $c \in [0, 1]$, $\mathcal{H} = L^2(0, 1)$, and $f \in C([0, 1])$. Show that $q[f] := |f(c)|^2$ is a well-defined, non-negative quadratic form that is not closable. (Hint: Consider the sequence $f_n(x) := \max\{0, 1 - n|x - c|\}$.)